The Genus Two Partition Function for Free Bosonic and Lattice Vertex Operator Algebras

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Abstract

We define the n-point function for a vertex operator algebra on a genus two Riemann surface in two separate sewing schemes where either two tori are sewn together or a handle is sewn to one torus. We explicitly obtain closed formulas for the genus two partition function for the Heisenberg free bosonic string and lattice vertex operator algebras in both sewing schemes. We prove that the partition functions are holomorphic in the sewing parameters on given suitable domains and describe their modular properties. Finally, we show that the partition functions cannot be equal in the neighborhood of a two-tori degeneration point where they can be explicitly compared.

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1 Introduction

One of the most striking features of conformal field theory (aka the theory of vertex algebras) is the occurrence of *elliptic functions* and *modular forms*, manifested in the form of n-point correlation trace functions. This phenomenon has been present in the physics of strings since the earliest days e.g. [GSW], [P]. In mathematics it dates from the Conway-Norton conjectures [CN] proved by Borcherds ([B1], [B2]), and Zhu's important paper [Z1]. Geometrically, we are dealing with probability amplitudes corresponding to a complex torus (compact Riemann surface of genus one) inflicted with n punctures corresponding to local fields (vertex operators). For a Vertex Operator Algebra (VOA) $V = \oplus V_n$, the most familiar correlation function is the 0-point function, also called the *graded dimension* or *partition function*

$$Z_V(q) = q^{-c/24} \sum_n \dim V_n q^n \tag{1}$$

(c is the central charge). An example which motivates much of the present paper is that of a lattice theory V_L associated to a positive-definite even lattice L. Then c is the rank of L and

$$Z_{V_L}(q) = \frac{\theta_L(q)}{q^{c/24}\prod_n (1-q^n)^c} \tag{2}$$

where $\theta_L(q)$ is the usual theta function of L and the denominator is the cth power of the Dedekind eta function $\eta(q)$. Both of these functions are (holomorphic) elliptic modular forms of weight $c/2$ on a certain congruence subgroup of $SL(2,\mathbb{Z})$, so that Z_{V_L} is an elliptic modular function of weight zero on the same subgroup. It is widely expected that an analogous result holds for *any rational* vertex operator algebra, namely that $Z_V(q)$ is a modular function of weight zero on a congruence subgroup of $SL(2,\mathbb{Z})$.

There are natural physical and mathematical reasons for wanting to extend this picture to Riemann surfaces of *higher genus*. In particular, we want to know if there are natural analogs of (1) and (2) for arbitrary rational vertex operator algebras and arbitrary genus, in which *genus g Siegel modular forms* occur. This is much more challenging than the case of genus one. Many, but not all, of the new difficulties that arise are already present at genus two, and it is this case that we are concerned with in the present paper. Our goal, then, is this: given a vertex operator algebra V, to introduce

the n-point correlation functions on a compact Riemann surface of genus two which are associated to V, and study their automorphic properties. An overview of aspects of this program is given in the Introduction to [MT2]. Brief discussions of some of our methods and results can also be found in [T] and [MT3].

Generally, our approach to genus two correlation functions is to define them in terms of genus one data coming from V. Now there are two rather different ways to obtain a compact Riemann surface of genus two from surfaces of genus one - one may sew two separate tori together, or self-sew a torus (i.e. attach a handle). We refer to these two schemes as the ϵ- and ρ-formalism respectively. As a result, much of our discussion is bifurcated as we are obliged to treat the two cases separately. The present paper depends heavily on results obtained in [MT2], reviewed in Section 2 below, concerned with a pair of maps

$$\mathcal{D}^{\epsilon} \xrightarrow{F^{\epsilon}} \mathbb{H}_2 \xleftarrow{F^{\rho}} \mathcal{D}^{\rho} \tag{3}$$

For $g \geq 1, \mathbb{H}_g$ is the genus g Siegel upper half-space. Then $\mathcal{D}^{\epsilon} \subseteq \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C}$ is the domain consisting of triples $(\tau_1, \tau_2, \epsilon)$ which correspond to a pair of complex tori of modulus τ_1, τ_2 sewn together according to the relation $z_1 z_2 = \epsilon$. (For more details, see [MT2] and Section 2 below.) $\mathcal{D}^{\rho} \subseteq \mathbb{H}_1 \times \mathbb{C}^2$ is similarly defined in terms of data (τ, w, ρ) needed to self-sew a torus of modulus τ. In each case, sewing produces a compact Riemann surface of genus two, and the maps $F^{\bullet}$ are those which assign to a point of $\mathcal{D}^{\bullet}$ the period matrix Ω of the sewn surface. Thus $F^{\bullet}$ takes values in $\mathbb{H}_2$, and in [MT2] we established that the $F^{\bullet}$ are holomorphic. (Here and below, it is sometimes convenient to use a bullet in place of subscripts and superscripts when comparing the ϵ- and ρ-formalisms).

In Section 3 we introduce some graph-theoretic technology which provides a convenient way of describing the period matrix Ω in the ϵ- and ρ-formalisms. Similar graphical techniques are employed later on as means of computing the genus two partition function for free bosonic (Heisenberg) and lattice VOAs. Section 4 is a brief review of some necessary background on VOA theory and the Li-Zamalodchikov or Li-Z metric. We assume throughout that the Li-Z metric is unique and invertible (which follows if V is simple [Li]).

Section 5 develops a theory of n-point functions for VOAs on Riemann surfaces of genus 0, 1 and 2 motivated by ideas in conformal field theory

([FS], [So1], [P]). The Zhu theory [Z1] of genus one n-point functions is reformulated in this language in terms of the self-sewing a Riemann sphere to obtain a torus. We consider two examples of such a sewing procedure where an interesting application of the Catalan series from combinatorics arises. We next consider two separate definitions of genus two n-point functions based on the ϵ- and ρ-formalisms. The genus two partition functions involve extending (3) to a diagram

$$\begin{array}{ccccc} \mathcal{D}^{\epsilon} & \xrightarrow{F^{\epsilon}} & \mathbb{H}_2 & \xleftarrow{F^{\rho}} & \mathcal{D}^{\rho} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{C} & & \end{array}$$

where the partition functions are maps $\mathcal{D}^{\bullet} \to \mathbb{C}$, defined purely in terms of genus one data coming from V. Explicitly, the genus two partition function of V in the two formalisms are as follows:

$$Z^{(2)}_{V,\epsilon}(\tau_1, \tau_2, \epsilon) = \sum_{n\geq 0} \epsilon^n \sum_{u\in V_{[n]}} Z^{(1)}_V(u, \tau_1) Z^{(1)}_V(\bar{u}, \tau_2), \tag{4}$$

$$Z^{(2)}_{V,\rho}(\tau, w, \rho) = \sum_{n\geq 0} \rho^n \sum_{u\in V_{[n]}} Z^{(1)}_V(\bar{u}, u, w, \tau). \tag{5}$$

Here, $Z^{(1)}_V(u, \tau)$ and $Z^{(1)}_V(\bar{u}, u, w, \tau)$ are (genus one) 1- and 2-point functions respectively with $\bar{u}$ the Li-Z metric dual of u. The precise meaning of (4) and (5), together with similar definitions for n-point functions, is given in Section 5. In any case, at genus two we get not one, but two, rather differently defined partition functions of V.

In Sections 6-9 we investigate the case of free bosonic theories (i.e. the Heisenberg vertex operator algebra M^c corresponding to c free bosons) and lattice theories in depth. The definitions of the genus two partition functions are *a priori* just formal power series in variables ϵ, q_1, q_2 and ρ, w, q respectively (as usual, $q = e^{2\pi i\tau}$, etc.). However, we will see, at least in the case of M^c and V_L, that they are in fact holomorphic functions on $\mathcal{D}^{\bullet}$. It is natural to expect that this result holds in much wider generality. Section 6 is devoted to the case of M^c in the ϵ-formalism. In this case, holomorphy depends on an interesting new formula for the genus two partition function. Namely, we prove (Theorem 6.1) by reinterpreting (4) in terms of certain graphical expansion, that

$$Z^{(2)}_{M^c,\epsilon}(\tau_1, \tau_2, \epsilon) = \frac{Z^{(1)}_{M^c}(\tau_1) Z^{(1)}_{M^c}(\tau_2)}{\det(I - A_1 A_2)^{c/2}}. \tag{6}$$

Here, the A_i are certain infinite matrices whose entries are expressions involving quasi-modular forms, and $Z^{(1)}_{M^c}(\tau_i) = q_i^{c/24}/\eta(q_i)$ the corresponding genus one partition function[1]. The matrices A_i and the infinite determinant that occurs in (6) were introduced and discussed at length in [MT2]. The results obtained there are important here, as are the explicit computations of 1-point functions obtained in [MT1]. We also give in Section 6 a product formula for the infinite determinant (Theorem 6.6) which depends on the graphical interpretation of the entries of the A_i.

The domain $\mathcal{D}^\epsilon$ admits the group $G_0 = SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ as automorphisms (in fact, there is a larger automorphism group G that contains G_0 with index 2.) We show (cf. Theorem 6.8) that the modular partition function $Z^{(2)}_{M^c,\mathrm{mod}}(\tau_1,\tau_2,\epsilon) \equiv Z^{(2)}_{M^c,\epsilon} q_1^{-c/24} q_2^{-c/24}$ is an automorphic form of weight $-c/2$ on G. This is a bit imprecise in several ways: we have not explained here what the automorphy factor is, and in fact this is an interesting point because it depends on the map F^ϵ. And similarly to the eta-function, there is a 24th root of unity, corresponding to a character of G, that intervenes in the functional equation. In any case if, for example, the central charge is 26 (the case of relevance to the critical bosonic string), $Z^{(2)}_{M^{26},\mathrm{mod}}$ is an automorphic form of weight -13 on G transforming according to an explicit character of G of order 12. These properties of $Z^{(2)}_{M^c,\mathrm{mod}}(\tau_1,\tau_2,\epsilon)$ justify the idea that it should be thought of as the genus two analog (in the ϵ-formalism) of $\eta(q)^{-c}$.

Section 8 is devoted to a reprise of the contents of Section 6 in the ρ-formalism. This is not merely a simple translation, however, and there are several issues in the ρ-formalism that need additional attention and which generally make the ρ-formalism more complicated than its ϵ-counterpart. This circumstance was already evident in [MT2], and stems in part from the fact that the map F^ρ involves a logarithmic term that is absent in the ϵ-formalism. The A-matrices are also more unwieldy, as they must be considered as having entries which themselves are 2×2 block-matrices with entries which are elliptic-type functions. Be that as it may, we establish in Theorem 8.5 the analog of (6) for $Z^{(2)}_{M^c,\rho}(\tau,w,\rho)$ in the ρ-formalism, a product formula (Theorem 8.6) for the intervening infinite determinant, and automorphy of $Z^{(2)}_{M^c,\mathrm{mod}}(\tau,w,\rho) \equiv Z^{(2)}_{M^c,\rho} q^{-c/24}$ with respect to a group $\Gamma_1 \cong SL(\mathbb{Z})$ (Theorem 8.8). In particular we obtain the holomorphy of the partition function.

In view of these strong formal similarities between the two partition func-

[1]In (4), (5) there are no overall factors $q_i^{-c/24}$, whereas in (2) there is.

tions $Z^{(2)}_{M^c,\epsilon}(\tau_1,\tau_2,\epsilon)$ and $Z^{(2)}_{M^c,\rho}(\tau,w,\rho)$ associated to 2 free bosons, it is natural to ask if they are equal in some sense. One could ask the same question for rational theories V too. In the very special case in which V is holomorphic (i.e. it has a *unique* irreducible module) one knows (e.g. [TUY]) that the genus 2 conformal block is 1-dimensional, in which case an identification of the two partition functions might seem inevitable. Of course, the partition functions are defined on different domains, so there is no question of them being literally equal. What one could (and should) ask is the following: is there a holomorphic map $f : \mathbb{H}_2 \to \mathbb{C}$ which makes the following diagram commute?

$$\begin{array}{ccccc} D^{\epsilon} & \xrightarrow{F^{\epsilon}} & \mathbb{H}_2 & \xleftarrow{F^{\rho}} & D^{\rho} \\ & \overset{Z^{(2)}_{\epsilon}}{\searrow} & \downarrow f & \overset{Z^{(2)}_{\rho}}{\swarrow} & \\ & & \mathbb{C} & & \end{array} \tag{7}$$

One should also expect f to be automorphic in a way that is consistent with the automorphic properties of the partition functions. Prior comments notwithstanding, we will explain in Section 10 that there is *no* such f, even for holomorphic theories such as V_L (where the lattice L is self-dual). How can we understand this situation? In the holomorphic case, say, how can it be reconciled with the uniqueness of the conformal block?

In Sections 7 and 9 we elucidate the expressions (4) and (5), thereby obtaining some of the main results of the present paper. Taking L of rank c, we prove that

$$\frac{Z^{(2)}_{V_L,\epsilon}(\tau_1,\tau_2,\epsilon)}{Z^{(2)}_{M^c,\epsilon}(\tau_1,\tau_2,\epsilon)} = \frac{Z^{(2)}_{V_L,\rho}(\tau,w,\rho)}{Z^{(2)}_{M^c,\rho}(\tau,w,\rho)} = \theta^{(2)}_L(\Omega). \tag{8}$$

Here, $\theta^{(2)}_L(\Omega)$ is the genus two Siegel theta function attached to L ([F]), and the equalities hold (even as formal power series) when Ω is interpreted as $F^{\epsilon}(\tau_1,\tau_2,\epsilon)$ or $F^{\rho}(\tau,w,\rho)$ according to the sewing scheme. Inasmuch as the genus one partition function for c free bosons is $\eta(q)^{-c}$, we see that (2) is just the genus one analog of (8).

Call the displayed quotients of partition functions the *normalized* partition function. Then (8) says exactly that (7) holds (with f being the Siegel

theta function) if we replace the partition functions with their normalized versions. Thus the following diagram commutes

$$\begin{array}{ccccc} D^{\epsilon} & \stackrel{F^{\epsilon}}{\longrightarrow} & \mathbb{H}_2 & \stackrel{F^{\rho}}{\longleftarrow} & D^{\rho} \\ & \stackrel{\hat{Z}^{(2)}_{\epsilon}}{\searrow} & \downarrow \theta^{(2)}_L & \stackrel{\hat{Z}^{(2)}_{\rho}}{\swarrow} & \\ & & \mathbb{C} & & \end{array}$$

We can put it this way: the *normalized* partition functions are *independent of the sewing scheme*. They can be identified, via the sewing maps $F^{\bullet}$, with *a holomorphic genus two Siegel modular form of weight $c/2$ - the Siegel theta function in this case*. It is the normalized partition function which can be appropriately identified with an element of the conformal block. We conjecture that analogous results hold for *any* rational vertex operator algebra and *any* genus. Section 11 contains a brief further discussion of these issues in the light of related ideas in string theory and algebraic geometry. That the two normalized partition functions turn out to be equal is, from our current perspective, something of a miracle. It would be very useful to have available an *a priori* proof of this circumstance.

2 Genus Two Riemann Surfaces from Sewn Tori

In this section we review the main results of [MT2] relevant to the present work. We review two separate constructions of a genus two Riemann surface based on a general sewing formalism due to Yamada [Y]. In the first construction, which we refer to as the ϵ-formalism, we parameterize a genus two Riemann surface by sewing together two once-punctured tori. In the second construction, which we refer to as the ρ-formalism, we parameterize a genus two Riemann surface by self-sewing a twice-punctured torus. In both cases, the period matrix is described by an explicit formula which defines a holomorphic map from a specified domain into the genus two Siegel upper half plane $\mathbb{H}_2$. In each case, this map is equivariant under a suitable subgroup of $Sp(4,\mathbb{Z})$. We also review the convergence and holomorphy of some infinite determinants that naturally arise and which play a dominant rôle later on.

2.1 Some Elliptic Function Theory

We begin with the definition of various modular and elliptic functions that permeate this work [MT1], [MT2]. We define

$$\begin{aligned} P_2(\tau,z) &= \wp(\tau,z)+E_2(\tau) \\ &= \frac{1}{z^2}+\sum_{k=2}^{\infty}(k-1)E_k(\tau)z^{k-2}, \end{aligned} \tag{9}$$

where $\tau\in\mathbb{H}_1$, the complex upper half-plane and where $\wp(\tau,z)$ is the Weierstrass function and $E_k(\tau)$ is equal to 0 for k odd, and for k even is the Eisenstein series

$$E_k(\tau)=E_k(q)=-\frac{B_k}{k!}+\frac{2}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \tag{10}$$

Here and below, we take $q=\exp(2\pi i\tau)$; $\sigma_{k-1}(n)=\sum_{d|n}d^{k-1}$, and B_k is a kth Bernoulli number e.g. [Se]. If $k\geq 4$ then $E_k(\tau)$ is a holomorphic modular form of weight k on $SL(2,\mathbb{Z})$ whereas $E_2(\tau)$ is a quasi-modular form [KZ],[MT2]. We define $P_0(\tau,z)$, up to a choice of the logarithmic branch, and $P_1(\tau,z)$ by

$$P_0(\tau,z) = -\log(z)+\sum_{k\geq 2}\frac{1}{k}E_k(\tau)z^k, \tag{11}$$

$$P_1(\tau,z) = \frac{1}{z}-\sum_{k\geq 2}E_k(\tau)z^{k-1}, \tag{12}$$

where $P_2=-\frac{d}{dz}P_1$ and $P_1=-\frac{d}{dz}P_0$. P_0 is related to the elliptic prime form $K(\tau,z)$ by [Mu1]

$$K(\tau,z)=\exp(-P_0(\tau,z)). \tag{13}$$

Define elliptic functions $P_k(\tau,z)$ for $k\geq 3$

$$P_k(\tau,z)=\frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}P_1(\tau,z). \tag{14}$$

Define for $k,l\geq 1$

$$C(k,l) = C(k,l,\tau)=(-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}E_{k+l}(\tau), \tag{15}$$

$$D(k,l,z) = D(k,l,\tau,z)=(-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}P_{k+l}(\tau,z). \tag{16}$$

Note that $C(k,l) = C(l,k)$ and $D(k,l,z) = (-1)^{k+l}D(l,k,z)$. We also define for $k \geq 1$,

$$\begin{aligned} C(k,0) &= C(k,0,\tau) = (-1)^{k+1}E_k(\tau), & (17) \\ D(k,0,z,\tau) &= (-1)^{k+1}P_k(z,\tau). & (18) \end{aligned}$$

The Dedekind eta-function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{19}$$

2.2 The ϵ-Formalism for Sewing Two Tori

We now review a general method due to Yamada [Y] and discussed at length in [MT2] for calculating the period matrix Ω of the genus two Riemann surface formed by sewing together two tori $\mathcal{S}_a$ for $a = 1, 2$. We shall sometimes refer to $\mathcal{S}_1$ and $\mathcal{S}_2$ as the left and right torus respectively. Consider an oriented torus $\mathcal{S}_a = \mathbb{C}/\Lambda_a$ with lattice $\Lambda_a = 2\pi i(\mathbb{Z}\tau_a \oplus \mathbb{Z})$ for $\tau_a \in \mathbb{H}_1$. For local coordinate $z_a \in \mathbb{C}/\Lambda_a$ consider the closed disk $|z_a| \leq r_a$ which is contained in $\mathcal{S}_a$ provided $r_a < \frac{1}{2}D(q_a)$ where

$$D(q_a) = \min_{\lambda\in\Lambda_a,\lambda\neq 0}|\lambda|,$$

is the minimal lattice distance. Introduce a complex sewing parameter ϵ where $|\epsilon| \leq r_1 r_2 < \frac{1}{4}D(q_1)D(q_2)$ and excise the disk $\{z_a, |z_a| \leq |\epsilon|/r_{\bar{a}}\}$ centred at $z_a = 0$ where we use the convention

$$\overline{1} = 2, \quad \overline{2} = 1. \tag{20}$$

Defining the annulus

$$\mathcal{A}_a = \{z_a, |\epsilon|/r_{\bar{a}} \leq |z_a| \leq r_a\}, \tag{21}$$

we identify $\mathcal{A}_1$ with $\mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{22}$$

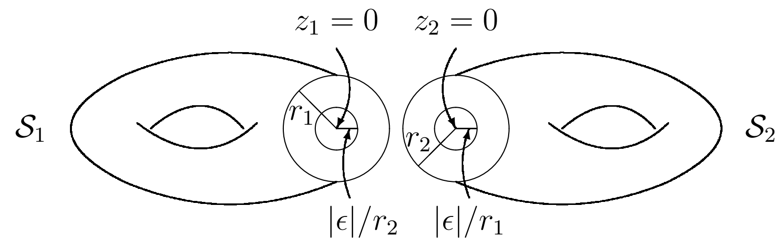

Fig. 1 Sewing Two Tori

The genus two Riemann surface is then parameterized by the domain

$$\mathcal{D}^{\epsilon} = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(q_1) D(q_2)\}. \tag{23}$$

The genus two period matrix $\Omega \in \mathbb{H}_2$, the Siegel upper half plane, may be determined as a function of $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^{\epsilon}$ in terms of certain infinite dimensional moment matrices $A_a(\tau_a, \epsilon) = (A_a(k, l, \tau_a, \epsilon))$ for $k, l \geq 1$ where

$$A_a(k, l, \tau_a, \epsilon) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} C(k, l, \tau_a). \tag{24}$$

These matrices play a dominant role both in the description of Ω and in our later discussion of the free bosonic and lattice VOA genus two partition functions. In particular, the matrix $I - A_1 A_2$ and $\det(I - A_1 A_2)$ (where I denotes the infinite identity matrix) play an important role where $\det(I - A_1 A_2)$ is defined by

$$\begin{aligned} \log \det(I - A_1 A_2) &= tr \log(I - A_1 A_2) \\ &= -\sum_{n \geq 1} \frac{1}{n} tr((A_1 A_2)^n). \end{aligned} \tag{25}$$

One finds

Theorem 2.1
(a) (op. cite., Proposition 1) The infinite matrix

$$(I - A_1 A_2)^{-1} = \sum_{n \geq 0} (A_1 A_2)^n, \tag{26}$$

is convergent for $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^{\epsilon}$.
(b) (op. cite., Theorem 2 & Proposition 3) $\det(I - A_1 A_2)$ *is non-vanishing and holomorphic for* $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^{\epsilon}$. $\square$

Furthermore, the genus two period matrix Ω is given by:

Theorem 2.2 *(op. cite., Theorem 4) The ϵ-formalism determines a holomorphic map*

$$\begin{aligned} F^\epsilon : \mathcal{D}^\epsilon &\rightarrow \mathbb{H}_2, \\ (\tau_1, \tau_2, \epsilon) &\mapsto \Omega(\tau_1, \tau_2, \epsilon), \end{aligned} \tag{27}$$

where $\Omega = \Omega(\tau_1, \tau_2, \epsilon)$ is given by

$$2\pi i \Omega_{11} = 2\pi i \tau_1 + \epsilon (A_2 (I - A_1 A_2)^{-1})(1,1), \tag{28}$$

$$2\pi i \Omega_{22} = 2\pi i \tau_2 + \epsilon (A_1 (I - A_2 A_1)^{-1})(1,1), \tag{29}$$

$$2\pi i \Omega_{12} = -\epsilon (I - A_1 A_2)^{-1}(1,1). \tag{30}$$

Here $(1,1)$ refers to the $(1,1)$-entry of a matrix. $\square$

$\mathcal{D}^\epsilon$ is preserved under the action of $G \simeq (SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})) \rtimes \mathbb{Z}_2$, the direct product of two copies of $SL(2,\mathbb{Z})$ (the left and right torus modular groups) which are interchanged upon conjugation by an involution β as follows

$$\begin{aligned} \gamma_1.(\tau_1, \tau_2, \epsilon) &= \left(\frac{a_1\tau_1 + b_1}{c_1\tau_1 + d_1}, \tau_2, \frac{\epsilon}{c_1\tau_1 + d_1}\right), \\ \gamma_2.(\tau_1, \tau_2, \epsilon) &= \left(\tau_1, \frac{a_2\tau_2 + b_2}{c_2\tau_2 + d_2}, \frac{\epsilon}{c_2\tau_2 + d_2}\right), \\ \beta.(\tau_1, \tau_2, \epsilon) &= (\tau_2, \tau_1, \epsilon), \end{aligned} \tag{31}$$

for $(\gamma_1, \gamma_2) \in SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. There is a natural injection $G \rightarrow Sp(4,\mathbb{Z})$ in which the two $SL(2,\mathbb{Z})$ subgroups are mapped to

$$\Gamma_1 = \left\{ \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \ \Gamma_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix} \right\}, \tag{32}$$

and the involution is mapped to

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{33}$$

Thus as a subgroup of $Sp(4,\mathbb{Z})$, G also has a natural action on the Siegel upper half plane $\mathbb{H}_2$ where for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4,\mathbb{Z})$

$$\gamma.\Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{34}$$

One then finds

Theorem 2.3 *(op. cit., Theorem 5) F^ϵ is equivariant with respect to the action of G i.e. there is a commutative diagram for $\gamma \in G$,*

$$\begin{array}{ccc} \mathcal{D}^\epsilon & \stackrel{F^\epsilon}{\rightarrow} & \mathbb{H}_2 \\ \gamma\downarrow & & \downarrow\gamma \\ \mathcal{D}^\epsilon & \stackrel{F^\epsilon}{\rightarrow} & \mathbb{H}_2 \end{array}$$

$\square$

The limit $\epsilon \to 0$ corresponds (by construction) to a two tori degeneration point of the genus two surface where $\Omega \to \mathrm{diag}(\Omega_{11} = \tau_1, \Omega_{22} = \tau_2)$. One can then show that there is a G-invariant neighborhood of each degeneration point on which F^ϵ is invertible (op.cite., Proposition 4.10).

2.3 The ρ-Formalism for Self-Sewing a Torus

We may alternatively construct a genus two Riemann surface by self-sewing a twice-punctured torus. Consider an oriented torus $\mathcal{S} = \mathbb{C}/\Lambda$ with local coordinate z for lattice $\Lambda = 2\pi i(\mathbb{Z}\tau \oplus \mathbb{Z})$ with $\tau \in \mathbb{H}_1$. Consider two disks centred at $z = 0$ and $z = w$ with local coordinates $z_1 = z$ and $z_2 = z - w$ of radius $r_a < \frac{1}{2}D(q)$ for $a = 1, 2$. Note that r_1, r_2 must also be sufficiently small to ensure that the disks do not intersect on $\mathcal{S}$. Introduce a complex parameter ρ where $|\rho| \leq r_1 r_2$ and define annular regions $\mathcal{A}_a = \{z_a, |\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a\}$. We identify $\mathcal{A}_1$ with $\mathcal{A}_2$ as a single region via the sewing relation

$$z_1 z_2 = \rho. \tag{35}$$

The genus two Riemann surface (excluding the degeneration point $\rho = 0$) so constructed is then parameterized by the domain

$$\mathcal{D}^\rho = \{(\tau, w, \rho) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} | \quad |w - \lambda| > 2|\rho|^{1/2} > 0 \text{ for all } \lambda \in \Lambda\}, \tag{36}$$

where the first inequality follows from the requirement that the annuli do not intersect.

In the ρ-formalism the genus two period matrix is expressed as a function of $(\tau, w, \rho) \in \mathcal{D}^\rho$ in terms of a doubly-indexed infinite matrix $R(\tau, w, \rho) = (R_{ab}(k, l, \tau, w, \rho))$ for $k, l \geq 1$ and $a, b \in \{1, 2\}$ where [MT2]

$$R(k, l, \tau, w, \rho) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \left[\begin{array}{cc} D(k, l, \tau, w) & C(k, l, \tau) \\ C(k, l, \tau) & D(l, k, \tau, w) \end{array} \right]. \tag{37}$$

Note that $R_{ab}(k, l) = R_{\bar{a}\bar{b}}(l, k)$. Similarly to Theorem 2.1 we find $I - R$ and $\det(I - R)$ play a central role (where now I denotes the doubly-indexed identity matrix)

Theorem 2.4 *(a) (op. cite., Proposition 6) The infinite matrix*

$$(I - R)^{-1} = \sum_{n \geq 0} R^n, \tag{38}$$

is convergent for $(\tau, w, \rho) \in \mathcal{D}^\rho$.

(b) (op. cite., Theorem 7) $\det(I - R)$ *is non-vanishing and holomorphic for* $(\tau, w, \rho) \in \mathcal{D}^\rho$. $\square$

The period matrix Ω is determined as follows:

Theorem 2.5 *(op. cite., Proposition 11) The ρ-formalism determines a holomorphic map*

$$\begin{aligned} F^\rho : \mathcal{D}^\rho &\rightarrow \mathbb{H}_2, \\ (\tau, w, \rho) &\mapsto \Omega(\tau, w, \rho), \end{aligned} \tag{39}$$

where $\Omega = \Omega(\tau, w, \rho)$ *is given by*

$$2\pi i \Omega_{11} = 2\pi i \tau - \rho\sigma((I - R)^{-1}(1, 1)), \tag{40}$$

$$2\pi i \Omega_{12} = w - \rho^{1/2}\sigma((b(I - R)^{-1}(1)), \tag{41}$$

$$2\pi i \Omega_{22} = \log(-\frac{\rho}{K(\tau, w)^2}) - b(I - R)^{-1}\bar{b}^T. \tag{42}$$

K is the elliptic prime form, $b = (b_a(k, \tau, w, \rho))$ is a doubly-indexed infinite row vector[2]

$$b(k, \tau, w, \rho) = \frac{\rho^{k/2}}{\sqrt{k}}(P_k(\tau, w) - E_k(\tau))[-1, (-1)^k], \tag{43}$$

[2]Note that b is denoted by β in ref. [MT2].

with $\bar{b}_a = b_{\bar{a}}$. $(1,1)$ and (1) refer to the $(k,l)=(1,1)$, respectively, $(k)=(1)$ entries of an infinite matrix and row vector respectively. $\sigma(M)$ denotes the sum over the finite indices for a given 2×2 or 1×2 matrix M. $\square$

The domain $\mathcal{D}^\rho$ is preserved by the Jacobi group $J = SL(2,\mathbb{Z})\ltimes\mathbb{Z}^2$ where

$$\begin{aligned}(a,b).(\tau,w,\rho) &= (\tau, w+2\pi i a\tau + 2\pi i b, \rho), \quad (a,b)\in\mathbb{Z}^2 && (44)\\ \gamma_1.(\tau,w,\rho) &= \left(\frac{a_1\tau+b_1}{c_1\tau+d_1}, \frac{w}{c_1\tau+d_1}, \frac{\rho}{(c_1\tau+d_1)^2}\right), \quad \gamma_1\in\Gamma_1, && (45)\end{aligned}$$

with $\Gamma_1 = SL(2,\mathbb{Z})$. However, due to the branch structure of the logarithmic term in (42) the map F^ρ is not equivariant with respect to J. (Instead one must pass to a simply-connected covering space $\hat{\mathcal{D}}^\rho$ on which $L=\hat{H}\Gamma_1$, a split extension of $SL(2,\mathbb{Z})$ by an integer Heisenberg group $\hat{H}$, acts - see Section 6.3 of ref. [MT2] for details). However, on restricting to the modular subgroup Γ_1 we find

Theorem 2.6 *(op. cit., Theorem 11, Corollary 2) F^ρ is equivariant with respect to the action of Γ_1 i.e. there is a commutative diagram for $\gamma_1\in\Gamma_1$,*

$$\begin{array}{ccc} \mathcal{D}^\rho & \xrightarrow{F^\rho} & \mathbb{H}_2 \\ \gamma_1\downarrow & & \downarrow\gamma_1 \\ \mathcal{D}^\rho & \xrightarrow{F^\rho} & \mathbb{H}_2 \end{array},$$

where the action of Γ_1 on $\mathbb{H}_2$ is that of (32). $\square$

The two tori degeneration limit can also be considered in the ρ-formalism as follows. Define the Γ_1-invariant

$$\chi = -\frac{\rho}{w^2}. \tag{46}$$

Then one finds that two tori degeneration limit is given by $\rho, w\to 0$ for fixed χ where

$$\Omega \to \begin{pmatrix} \tau & 0 \\ 0 & \frac{1}{2\pi i}\log(f(\chi)) \end{pmatrix} \tag{47}$$

with $f(\chi)$ the Catalan series

$$f(\chi) = \frac{1-\sqrt{1-4\chi}}{2\chi} - 1 = \sum_{n\geq 1}\frac{1}{n}\binom{2n}{n+1}\chi^n. \tag{48}$$

(The coefficients $\frac{1}{n}\binom{2n}{n+1}$ are the ubiquitous Catalan numbers of combinatorics e.g. [St].)

In order to describe the limit (47) more precisely, we introduce the domain

$$\mathcal{D}^{\chi} = \{(\tau, w, \chi) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} \mid (\tau, w, -w^2\chi) \in \mathcal{D}^{\rho}, 0 < |\chi| < \frac{1}{4}\}, \qquad (49)$$

and a Γ_1-equivariant holomorphic map

$$\begin{aligned} F^{\chi} : \mathcal{D}^{\chi} &\rightarrow \mathbb{H}_2, \\ (\tau, w, \chi) &\mapsto \Omega^{(2)}(\tau, w, -w^2\chi). \end{aligned} \qquad (50)$$

Then

$$\mathcal{D}_0^{\chi} = \{(\tau, 0, \chi) \in \mathbb{H}_1 \times \mathbb{C} \times \mathbb{C} | 0 < |\chi| < \frac{1}{4}\},$$

is the space of two-tori degeneration limit points of the domain $\mathcal{D}^{\chi}$. We may then compare the two parameterizations on certain Γ_1-invariant neighborhoods of a two tori degeneration point in both parameterizations to obtain

Theorem 2.7 *(op. cit., Theorem 12) There exists a 1-1 holomorphic mapping between Γ_1-invariant open domains $\mathcal{I}^{\chi} \subset (\mathcal{D}^{\chi} \cup \mathcal{D}_0^{\chi})$ and $\mathcal{I}^{\epsilon} \subset \mathcal{D}^{\epsilon}$ where $\mathcal{I}^{\chi}$ and $\mathcal{I}^{\epsilon}$ are open neighborhoods of a two tori degeneration point.* $\square$

3 Graphical expansions

3.1 Rotationless and chequered cycles

We set up some notation and discuss certain types of labelled graphs. These arise directly from consideration of the terms that appear in the expressions for Ω_{ij} reviewed in the last Section, and will later play an important rôle in the analysis of genus two partition functions for vertex operator algebras.

Consider a set of independent (non-commuting) variables x_i indexed by the elements of a finite set $I = \{1, ..., N\}$. The set of all distinct monomials $x_{i_1}...x_{i_n} (n \geq 0)$ may be considered as a basis for the tensor algebra associated with an N dimensional vector space. Call n the degree of the monomial $x_{i_1}...x_{i_n}$.

Let $\rho = \rho_n$ be the standard cyclic permutation which acts on monomials of degree n via $\rho : x_{i_1}...x_{i_n} \mapsto x_{i_n}x_{i_1}...x_{i_{n-1}}$. The *rotation group* of a given

monomial $x = x_{i_1}...x_{i_n}$ is the subgroup of $\langle\rho_n\rangle$ that leaves x invariant. Call x *rotationless* in case its rotation group is trivial. Let us say that two monomials x, y of degree n are *equivalent* in case $y = \rho_n^r(x)$ for some $r \in Z$, and denote the corresponding equivalence class by (x). We call these *cycles*. Note that equivalent monomials have the same rotation group, so we may meaningfully refer to the rotation group of a cycle. In particular, a *rotationless cycle* is a cycle whose representative monomials are themselves rotationless. Let C_n be the set of inequivalent cycles of degree n.

It is convenient to identify a cycle $(x_{i_1}...x_{i_n})$ with a *cyclic labelled graph* or *labelled polygon*, that is, a graph with n vertices labelled $x_{i_1}, ..., x_{i_n}$ and with edges $x_{i_1}x_{i_2}, ..., x_{i_{n-1}}x_{i_n}, x_{i_n}x_{i_1}$. We will sometimes afflict the graph with one of the two canonical orientations.

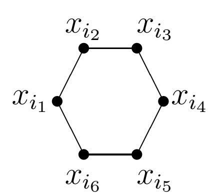

Fig. 2 A Cyclic Labelled Graph

A cycle is rotationless precisely when its graph admits no non-trivial rotations (a rotation now being an orientation-preserving automorphism of the graph which preserves labels of nodes).

Next we introduce the notion of a *chequered cycle* as a (clockwise) oriented, labelled polygon L with $2n$ nodes for some integer $n \geq 0$, and nodes labelled by arbitrary positive integers. Moreover, edges carry a label 1 or 2 which alternate as one moves around the polygon.

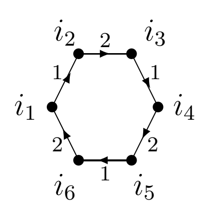

Fig. 3 Chequered Cycle

We call a node with label 1 *distinguished* if its abutting edges are of type $\xrightarrow{2} \overset{1}{\bullet} \xrightarrow{1}$. Set

$$\begin{aligned}
\mathcal{R} &= \{\text{isomorphism classes of rotationless chequered cycles }\}, \quad (51)\\
\mathcal{R}_{21} &= \{\text{isomorphism classes of rotationless chequered cycles}\\
&\quad \text{with a distinguished node}\},\\
\mathcal{L}_{21} &= \{\text{isomorphism classes of chequered cycles with a}\\
&\quad \textit{unique} \text{ distinguished node}\},
\end{aligned}$$

Let S be a commutative ring and $S[t]$ the polynomial ring with coefficients in S. Let M_1 and M_2 be infinite matrices with (k,l)-entries

$$M_a(k,l) = t^{k+l} s_a(k,l) \tag{52}$$

for $a = 1, 2$ and $k, l \geq 1$, where $s_a(k,l) \in S$. Given this data, we define a map, or *weight function*,

$$\omega : \{\text{chequered cycles}\} \longrightarrow S[t]$$

as follows: if L is a chequered cycle then L has edges E labelled as $\overset{k}{\bullet} \xrightarrow{a} \overset{l}{\bullet}$. Then set $\omega(E) = M_a(k,l)$ and

$$\omega(L) = \prod \omega(E) \tag{53}$$

where the product is taken over all edges of L.

It is useful to also introduce a variation on the theme of chequered polygons, namely *oriented chequered necklaces.* These are connected graphs with $n \geq 3$ nodes, $(n-2)$ of which have valency 2 and two of which have valency 1 (these latter are the *end nodes*) together with an orientation, say from left to right. There is also a degenerate necklace N_0 with a single node and no edges. As before, nodes are labelled with arbitrary positive integers and edges are labelled with an index 1 or 2 which alternate along the necklace. For such a necklace N, we define the weight function $\omega(N)$ as a product of edge weights as in (53), with $\omega(N_0) = 1$.

Among all chequered necklaces there is a distinguished set for which both end nodes are labelled by 1. There are four types of such chequered necklaces,

which may be further distinguished by the labels of the two edges at the extreme left and right. Using the convention (20) we say that the chequered necklace

$$\underset{1}{\bullet} \xrightarrow{\bar{a}} \underset{i}{\bullet} \ldots \underset{j}{\bullet} \xrightarrow{\bar{b}} \underset{1}{\bullet}$$

Fig. 4

is of *type ab* for $a, b \in \{1, 2\}$, and set

$$\begin{aligned} \mathcal{N}_{ab} &= \{\text{isomorphism classes of oriented chequered} \\ &\qquad \text{necklaces of type } ab\}, \end{aligned} \tag{54}$$

$$\omega_{ab} = \sum_{N \in \mathcal{N}_{ab}} \omega(N). \tag{55}$$

There are other types of graphs that we will need, adapted to the ρ-formalism. Rather than discussing them here, we delay their introduction until Subsection 3.3 below.

3.2 Graphical expansions for Ω in the ϵ-formalism

We now apply the formalism of the previous Subsection to the expressions for Ω_{ij} in the ϵ-formalism reviewed in Section 2. The ring S is taken to be the product $S_1 \times S_2$ where for $a = 1, 2$, S_a is the ring of quasi-modular forms $\mathbb{C}[E_2(\tau_a), E_4(\tau_a), E_6(\tau_a)]$, and $t = \epsilon^{1/2}$. The matrices M_a are taken to be the A_a defined in (24). Thus

$$s_a(k, l) = \frac{C(k, l, \tau_a)}{\sqrt{kl}}, \tag{56}$$

and for the edge E labelled as $\overset{k}{\bullet} \xrightarrow{a} \overset{l}{\bullet}$ we have

$$\omega(E) = A_a(k, l). \tag{57}$$

We can now state

Proposition 3.1 *In the ϵ−formalism we have*

$$\Omega_{12} = -\frac{\epsilon}{2\pi i} \prod_{L \in \mathcal{R}_{21}} (1 - \omega(L))^{-1}. \tag{58}$$

Proof. Beyond the intrinsic interest of this product formula, our main use of it will be to provide an alternate proof of Theorem 6.8 below. We therefore relegate the proof to Proposition 12.3 in an Appendix. $\square$

We will need some further identities of this nature. Recalling the notation (55),

Proposition 3.2 *([MT2], Proposition 4) In the ϵ-formalism we have for $a = 1, 2$ that*

$$\begin{aligned} \Omega_{aa} &= \tau_a + \frac{\epsilon}{2\pi i}\omega_{aa}, \\ \Omega_{a\bar{a}} &= -\frac{\epsilon}{2\pi i}\omega_{a\bar{a}}. \qquad \square \end{aligned}$$

Remark 3.3 *Proposition 3.1 implies that*

$$\omega_{12} = \prod_{L \in \mathcal{R}_{21}} (1 - \omega(L))^{-1}.$$

3.3 Graphical expansions for Ω in the ρ-formalism

We turn next to the expressions for Ω in the ρ-formalism reviewed in Section 2. In this case it is natural to introduce *doubly-indexed cycles.* These are (clockwise) oriented, labelled polygons L with n nodes for some integer $n \geq 1$, nodes being labelled by a pair of integers k, a where $k \geq 1$ and $a \in \{1, 2\}$. Thus, a typical doubly-indexed cycle looks as follows:

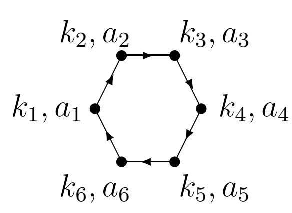

Fig. 5 Doubly-Indexed Cycle

Next we define a weight function ω with values in the ring of elliptic functions and quasi-modular forms $\mathbb{C}[P_2(\tau,w),P_3(\tau,w),E_2(\tau),E_4(\tau),E_6(\tau)]$ as follows: if L is a doubly-indexed cycle then L has edges E labelled as $\overset{k,a}{\bullet} \to \overset{l,b}{\bullet}$, and we set

$$\omega(E) = R_{ab}(k,l,\tau,w,\rho),$$

with $R_{ab}(k,l)$ as in (37) and

$$\omega(L) = \prod \omega(E).$$

As usual, the product is taken over all edges of L. It is straightforward (but of less value for our purposes) to obtain an analog of Proposition 12.3.

We also introduce *doubly-indexed necklaces* $\mathcal{N} = \{N\}$. These are connected graphs with $n \geq 2$ nodes, $(n-2)$ of which have valency 2 and two of which have valency 1 together with an orientation, say from left to right, on the edges. In this case, each vertex carries two integer labels k, a with $k \geq 1$ and $a \in \{1,2\}$. We define the degenerate necklace N_0 to be a single node with no edges, and set $\omega(N_0) = 1$.

We define necklaces with distinguished end nodes labelled $k,a;l,b$ as follows:

$$\underset{k,a}{\bullet} \longrightarrow \underset{k_1,a_1}{\bullet} \cdots \underset{k_2,a_2}{\bullet} \longrightarrow \underset{l,b}{\bullet} \qquad \text{(type } k,a;l,b)$$

and set

$$\mathcal{N}(k,a;l,b) = \{\text{isomorphism classes of necklaces of type } k,a;l,b\}.$$

It is convenient to define

$$\begin{aligned}
\omega_{11} &= \sum_{a_1,a_2=1,2} \sum_{N\in\mathcal{N}(1,a_1;1,a_2)} \omega(N), \\
\omega_{b1} &= \sum_{a_1,a_2=1,2} \sum_{k\geq 1} b_{a_1}(k) \sum_{N\in\mathcal{N}(k,a_1;1,a_2)} \omega(N), \\
\omega_{b\bar{b}} &= \sum_{a_1,a_2=1,2} \sum_{k,l\geq 1} b_{a_1}(k)\bar{b}_{a_2}(l) \sum_{N\in\mathcal{N}(k,a_1;l,a_2)} \omega(N).
\end{aligned} \tag{59}$$

Then we find

Proposition 3.4 *([MT2], Proposition 12) In the ρ-formalism we have*

$$\begin{aligned} 2\pi i\Omega_{11} &= 2\pi i\tau - \rho\omega_{11}, \\ 2\pi i\Omega_{12} &= w - \rho^{1/2}\omega_{b1}, \\ 2\pi i\Omega_{22} &= \log(-\frac{\rho}{K(\tau,w)^2}) - \omega_{b\bar{b}}. \qquad \square \end{aligned}$$

4 Vertex operator algebras and the Li-Zamalodchikov metric

4.1 Vertex operator algebras

We review some relevant aspects of vertex operator algebras ([FHL],[FLM], [Ka], [LL], [MN]). A vertex operator algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, a linear map $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$, for formal parameter z, and a pair of distinguished vectors (states): the vacuum $\mathbf{1} \in V_0$, and the conformal vector $\omega \in V_2$. For each state $v \in V$ the image under the Y map is the vertex operator

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}, \tag{60}$$

with modes $v(n) \in \mathrm{End}V$ where $\mathrm{Res}_{z=0}z^{-1}Y(v, z)\mathbf{1} = v(-1)\mathbf{1} = v$. Vertex operators satisfy the Jacobi identity or equivalently, operator locality or Borcherds's identity for the modes (loc. cit.).

The vertex operator for the conformal vector ω is defined as

$$Y(w, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

The modes $L(n)$ satisfy the Virasoro algebra of central charge c:

$$[L(m), L(n)] = (m - n)L(m + n) + (m^3 - m)\frac{c}{12}\delta_{m,-n}.$$

We define the homogeneous space of weight k to be $V_k = \{v \in V | L(0)v = kv\}$ where we write $wt(v) = k$ for v in V_k . Then as an operator on V we have

$$v(n) : V_m \to V_{m+k-n-1}.$$

In particular, the *zero mode* $o(v) = v(wt(v) - 1)$ is a linear operator on V_m. A state v is said to be *quasi-primary* if $L(1)v = 0$ and *primary* if additionally $L(2)v = 0$.

The subalgebra $\{L(-1), L(0), L(1)\}$ generates a natural action on vertex operators associated with $SL(2, \mathbb{C})$ Möbius transformations on z ([B1], [DGM], [FHL], [Ka]). In particular, we note the inversion $z \mapsto 1/z$ for which

$$Y(v, z) \mapsto Y^{\dagger}(v, z) = Y(e^{zL(1)}(-\frac{1}{z^2})^{L(0)}v, \frac{1}{z}). \tag{61}$$

$Y^{\dagger}(v, z)$ is the *adjoint* vertex operator [FHL]. Under the dilatation $z \mapsto az$ we have

$$Y(v, z) \mapsto a^{L(0)}Y(v, z)a^{-L(0)} = Y(a^{L(0)}v, az). \tag{62}$$

We also note ([BPZ],[Z2]) that under a general origin-preserving conformal map $z \mapsto w = \phi(z)$,

$$Y(v, z) \mapsto Y((\phi'(z))^{L(0)}v, w), \tag{63}$$

for any primary vector v.

We consider some particular VOAs, namely Heisenberg free boson and lattice VOAs. Consider an l-dimensional complex vector space (i.e., abelian Lie algebra) $\mathfrak{H}$ equipped with a non-degenerate, symmetric, bilinear form $(\, ,)$ and a distinguished orthonormal basis $a_1, a_2, ... a_l$. The corresponding affine Lie algebra is the Heisenberg Lie algebra $\hat{\mathfrak{H}} = \mathfrak{H} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$ with brackets $[k, \hat{\mathfrak{H}}] = 0$ and

$$[a_i \otimes t^m, a_j \otimes t^n] = m\delta_{i,j}\delta_{m,-n}k. \tag{64}$$

Corresponding to an element λ in the dual space $\mathfrak{H}^*$ we consider the Fock space defined by the induced (Verma) module

$$M^{(\lambda)} = U(\hat{\mathfrak{H}}) \otimes_{U(\mathfrak{H}\otimes\mathbb{C}[t]\oplus\mathbb{C}k)} \mathbb{C},$$

where $\mathbb{C}$ is the 1-dimensional space annihilated by $\mathfrak{H} \otimes t\mathbb{C}[t]$ and on which k acts as the identity and $\mathfrak{H} \otimes t^0$ via the character λ; U denotes the universal enveloping algebra. There is a canonical identification of linear spaces

$$M^{(\lambda)} = S(\mathfrak{H} \otimes t^{-1}\mathbb{C}[t^{-1}]),$$

where S denotes the (graded) symmetric algebra. The Heisenberg free boson VOA M^l corresponds to the case $\lambda = 0$. The Fock states

$$v = a_1(-1)^{e_1}.a_1(-2)^{e_2}....a_1(-n)^{e_n}....a_l(-1)^{f_1}.a_l(-2)^{f_2}...a_l(-p)^{f_p}.\mathbf{1}, \quad (65)$$

for non-negative integers $e_i, ..., f_j$ form a basis of M^l. The vacuum $\mathbf{1}$ is canonically identified with the identity of $M_0 = \mathbb{C}$, while the weight 1 subspace M_1 may be naturally identified with $\mathfrak{H}$. M^l is a simple VOA of central charge l.

Next we consider the case of a lattice vertex operator algebra V_L associated to a positive-definite even lattice L (cf. [B1], [FLM]). Thus L is a free abelian group of rank l equipped with a positive definite, integral bilinear form $(\ ,\ ) : L \otimes L \to \mathbb{Z}$ such that (α, α) is even for $\alpha \in L$. Let $\mathfrak{H}$ be the space $\mathbb{C} \otimes_{\mathbb{Z}} L$ equipped with the $\mathbb{C}$-linear extension of $(\ ,\ )$ to $\mathfrak{H} \otimes \mathfrak{H}$ and let M^l be the corresponding Heisenberg VOA. The Fock space of the lattice theory may be described by the linear space

$$V_L = M^l \otimes \mathbb{C}[L] = \sum_{\alpha \in L} M^l \otimes e^{\alpha}, \quad (66)$$

where $\mathbb{C}[L]$ denotes the group algebra of L with canonical basis e^{α}, $\alpha \in L$. M^l may be identified with the subspace $M^l \otimes e^0$ of V_L, in which case M^l is a subVOA of V_L and the rightmost equation of (66) then displays the decomposition of V_L into irreducible M^l-modules. V_L is a simple VOA of central charge l. Each $\mathbf{1} \otimes e^{\alpha} \in V_L$ is a primary state of weight $\frac{1}{2}(\alpha, \alpha)$ with vertex operator (loc. cit.)

$$\begin{aligned} Y(\mathbf{1} \otimes e^{\alpha}, z) &= Y_-(\mathbf{1} \otimes e^{\alpha}, z) Y_+(\mathbf{1} \otimes e^{\alpha}, z) e^{\alpha} z^{\alpha}, \\ Y_{\pm}(\mathbf{1} \otimes e^{\alpha}, z) &= \exp(\mp \sum_{n>0} \frac{\alpha(\pm n)}{n} z^{\mp n}). \end{aligned} \quad (67)$$

The operators $e^{\alpha} \in \mathbb{C}[L]$ obey

$$e^{\alpha} e^{\beta} = \epsilon(\alpha, \beta) e^{\alpha+\beta} \quad (68)$$

for 2-cocycle $\epsilon(\alpha, \beta)$ satisfying $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$.

4.2 The Li-Zamolodchikov metric

A bilinear form $\langle\ ,\rangle : V \times V \longrightarrow \mathbb{C}$ is called *invariant* in case the following identity holds for all $a, b, c \in V$ ([FHL]):

$$\langle Y(a,z)b, c\rangle = \langle b, Y^{\dagger}(a,z)c\rangle, \tag{69}$$

with $Y^{\dagger}(a,z)$ the adjoint operator (61).

Remark 4.1 *Note that*

$$\begin{aligned}
\langle u, v\rangle &= \mathrm{Res}_{w=0} w^{-1} \mathrm{Res}_{z=0} z^{-1} \langle Y(u,w)\mathbf{1}, Y(v,z)\mathbf{1}\rangle \\
&= \mathrm{Res}_{w=0} w^{-1} \mathrm{Res}_{z=0} z^{-1} \langle \mathbf{1}, Y^{\dagger}(u,w) Y(v,z)\mathbf{1}\rangle \\
&= \text{“}\langle \mathbf{1}, Y(u, z=\infty) Y(v, z=0)\mathbf{1}\rangle\text{”},
\end{aligned} \tag{70}$$

with $w = 1/z$, following (61). Thus the invariant bilinear form is equivalent to what is known as the (chiral) Zamolodchikov metric in Conformal Field Theory ([BPZ], [P]).

Generally a VOA may have no non-zero invariant bilinear form, but a result of Li [Li] guarantees that if V_0 is spanned by the vacuum vector $\mathbf{1}$, and V is self-dual in the sense that V is isomorphic to the contragredient module V' as a V-module, then V has a unique non-zero invariant bilinear form up to scalar. Note that $\langle\ ,\rangle$ is necessarily symmetric by a theorem of [FHL]. Furthermore if V is simple then such a form is necessarily non-degenerate. All of the VOAs that occur in this paper satisfy these conditions, so that if we normalize so that $\langle \mathbf{1}, \mathbf{1}\rangle = 1$ then $\langle\ ,\rangle$ is unique. We refer to this particular bilinear form as the *Li-Zamolodchikov metric* on V, or LiZ-metric for short.

Remark 4.2 *Uniqueness entails that the LiZ-metric on the tensor product $V_1 \otimes V_2$ of a pair of simple VOAs satisfying the appropriate conditions is just the tensor product of the LiZ metrics on V_1 and V_2.*

If a is a homogeneous, quasi-primary state, the component form of (69) reads

$$\langle a(n)u, v\rangle = (-1)^{wt(a)} \langle u, a(2wt(a) - n - 2)v\rangle. \tag{71}$$

In particular, since the conformal vector ω is quasi-primary of weight 2 we may take ω in place of a in (71) and obtain

$$\langle L(n)u, v\rangle = \langle u, L(-n)v\rangle. \tag{72}$$

The case $n = 0$ of (72) shows that the homogeneous spaces V_n, V_m are orthogonal if $n \neq m$. Taking $u = \mathbf{1}$ and using $a = a(-1)\mathbf{1}$ in (71) yields

$$\langle a, v\rangle = (-1)^{wt(a)}\langle \mathbf{1}, a(2wt(a) - 1)v\rangle, \tag{73}$$

for a quasi-primary, and this affords a practical way to compute the LiZ-metric.

Consider the rank one Heisenberg (free boson) VOA $M = M^1$ generated by a weight one state a with $(a, a) = 1$. Then $\langle a, a\rangle = -\langle \mathbf{1}, a(1)a(-1)\mathbf{1}\rangle = -1$. Using (64), it is straightforward to verify that in general the Fock basis consisting of vectors of the form

$$v = a(-1)^{e_1}...a(-p)^{e_p}.\mathbf{1}, \tag{74}$$

for non-negative integers $\{e_i\}$ is orthogonal with respect to the LiZ-metric, and that

$$\langle v, v\rangle = \prod_{1\leq i\leq p}(-i)^{e_i}e_i!. \tag{75}$$

This result generalizes in an obvious way for a rank l free boson VOA M^l with Fock basis (65) following Remark 4.2.

We consider next the lattice vertex operator algebra V_L for a positive-definite even lattice L. We take as our Fock basis the states $\{v \otimes e^\alpha\}$ where v is as in (65) and α ranges over the elements of L.

Lemma 4.3 *If $u, v \in M^l$ and $\alpha, \beta \in L$, then*

$$\begin{aligned}\langle u \otimes e^\alpha, v \otimes e^\beta\rangle &= \langle u, v\rangle\langle \mathbf{1} \otimes e^\alpha, \mathbf{1} \otimes e^\beta\rangle \\ &= (-1)^{\frac{1}{2}(\alpha,\alpha)}\epsilon(\alpha, -\alpha)\langle u, v\rangle\delta_{\alpha,-\beta}.\end{aligned}$$

Proof. It follows by successive applications of (71) that the first equality in the Lemma is true, and that it is therefore enough to prove it in the case that $u = v = \mathbf{1}$. We identify the primary vector $\mathbf{1} \otimes e^\alpha$ with e^α in the following. Apply (73) in this case to see that $\langle e^\alpha, e^\beta\rangle$ is given by

$$\begin{aligned}&(-1)^{\frac{1}{2}(\alpha,\alpha)}\langle \mathbf{1}, e^\alpha((a, a) - 1)e^\beta\rangle \\ &= (-1)^{\frac{1}{2}(\alpha,\alpha)}\mathrm{Res}_{z=0}z^{(a,a)-1}\langle \mathbf{1}, Y(e^\alpha, z)e^\beta\rangle \\ &= (-1)^{\frac{1}{2}(\alpha,\alpha)}\epsilon(\alpha, \beta)\mathrm{Res}_{z=0}z^{(\alpha,\beta)+(a,a)-1}\langle \mathbf{1}, Y_-(\mathbf{1} \otimes e^\alpha, z).\mathbf{1} \otimes e^{\alpha+\beta}\rangle.\end{aligned}$$

Unless $\alpha+\beta=0$, all states to the left inside the bracket $\langle\ ,\rangle$ on the previous line have positive weight, hence are orthogonal to $\mathbf{1}$. So $\langle e^{\alpha}, e^{\beta}\rangle = 0$ if $\alpha+\beta \neq 0$. In the contrary case, the exponential operator acting on the vacuum yields just the vacuum itself among weight zero states, and we get $\langle e^{\alpha}, e^{-\alpha}\rangle = (-1)^{\frac{1}{2}(\alpha,\alpha)}\epsilon(\alpha,-\alpha)$ in this case. This completes the proof of the Lemma. $\square$

Corollary 4.4 *We may choose the cocycle so that* $\epsilon(\alpha,-\alpha) = (-1)^{\frac{1}{2}(\alpha,\alpha)}$ *(see Appendix). In this case, we have*

$$\langle u\otimes e^{\alpha}, v\otimes e^{\beta}\rangle = \langle u,v\rangle\delta_{\alpha,-\beta}. \tag{76}$$

5 Partition and n-point functions for vertex operator algebras on a Riemann Surface

In this section we consider the partition and n-point functions for a VOA on a Riemann surface of genus zero, one or two. Our definitions are based on sewing schemes for the given Riemann surface in terms of one or more surfaces of lower genus and are motivated by ideas in conformal field theory especially [FS], [So1] and [P]. We assume throughout that V has a non-degenerate LiZ metric $\langle\ ,\rangle$. Then for any V basis $\{u^{(a)}\}$, we may define the *dual basis* $\{\bar{u}^{(a)}\}$ with respect to the LiZ metric where

$$\langle u^{(a)}, \bar{u}^{(b)}\rangle = \delta_{ab}. \tag{77}$$

5.1 Genus zero

We begin with the definition of the genus zero n-point function given by:

$$Z_V^{(0)}(v_1,z_1;\ldots v_n,z_n) = \langle \mathbf{1}, Y(v_1,z_1)\ldots Y(v_n,z_n)\mathbf{1}\rangle, \tag{78}$$

for $v_1,\ldots v_n \in V$. In particular, the genus zero partition (or 0-point) function is $Z_V^{(0)} = \langle \mathbf{1},\mathbf{1}\rangle = 1$. The genus zero n-point function is a rational function of $z_1,\ldots z_n$ (with possible poles at $z_i = 0$ and $z_i = z_j, i\neq j$). Thus we may consider $z_1,\ldots z_n \in \mathbb{C}\cup\{\infty\}$, the Riemann sphere, with $Z_V^{(0)}(v_1,z_1;\ldots;v_n,z_n)$ evaluated for $|z_1| > |z_2| > \ldots > |z_n|$ (e.g. [FHL], [Z2], [GG]). The n-point function has a canonical geometric interpretation for primary vectors

v_i of $L(0)$ weight $wt(v_i)$. Then $Z^{(0)}_V(v_1, z_1; \ldots; v_n, z_n)$ parameterizes a global meromorphic differential form on the Riemann sphere,

$$\mathcal{F}^{(0)}_V(v_1, \ldots v_n) = Z^{(0)}_V(v_1, z_1; \ldots; v_n, z_n) \prod_{1 \leq i \leq n} (dz_i)^{wt(v_i)}. \tag{79}$$

It follows from (63) that $\mathcal{F}^{(0)}_V$ is conformally invariant. This interpretation is the starting point of various algebraic-geometric approaches to n-point functions (*apart* from the partition or 0-point function) at higher genera (e.g. [TUY], [Z2]).

It is instructive to consider $\mathcal{F}^{(0)}_V$ in the context of a trivial sewing of two Riemann spheres parameterized by z_1 and z_2 to form another Riemann sphere as follows. For $r_a > 0, a = 1, 2$, and a complex parameter ϵ satisfying $|\epsilon| \leq r_1 r_2$, excise the open disks $|z_a| < |\epsilon| r_{\bar{a}}^{-1}$ (recall convention (20)) and identify the annular regions $r_a \geq |z_a| \geq |\epsilon| r_{\bar{a}}^{-1}$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{80}$$

Consider $Z^{(0)}_V(v_1, x_1; \ldots v_n, x_n)$ for quasi-primary v_i with $r_1 \geq |x_i| \geq |\epsilon| r_2^{-1}$ and let $y_i = \epsilon / x_i$. Then for $0 \leq k \leq n-1$ we find from (77) that

$$\begin{gathered} Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} = \\ \sum_{r \geq 0} \sum_{u \in V_r} \langle \bar{u}, Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \rangle u, \end{gathered} \tag{81}$$

where the inner sum is taken over any basis for V_r. Thus

$$\begin{gathered} Z^{(0)}_V(v_1, x_1; \ldots v_n, x_n) = \\ \sum_{r \geq 0} \sum_{u \in V_r} \langle \mathbf{1}, Y(v_1, x_1) \ldots Y(v_k, x_k) u \rangle \langle \bar{u}, Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \rangle. \end{gathered}$$

But

$$\langle \mathbf{1}, Y(v_1, x_1) \ldots Y(v_k, x_k) u \rangle = \mathrm{Res}_{z_1=0} z_1^{-1} Z^{(0)}_V(v_1, x_1; \ldots v_k, x_k; u, z_1),$$

and

$$\begin{aligned} & \langle \bar{u}, Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \rangle \\ = \; & \langle \mathbf{1}, Y^\dagger(v_n, x_n) \ldots Y^\dagger(v_{k+1}, x_{k+1}) \bar{u} \rangle \\ = \; & \langle \mathbf{1}, \epsilon^{L(0)} Y^\dagger(v_n, x_n) \epsilon^{-L(0)} \ldots \epsilon^{L(0)} Y^\dagger(v_{k+1}, x_{k+1}) \epsilon^{-L(0)} \epsilon^{L(0)} \bar{u} \rangle \\ = \; & \epsilon^r \mathrm{Res}_{z_2=0} z_2^{-1} Z^{(0)}_V(v_n, y_n; \ldots v_{k+1}, y_{k+1}; \bar{u}, z_2) \prod_{k+1 \leq i \leq n} (-\frac{\epsilon}{x_i^2})^{wt(v_i)}. \end{aligned}$$

The last equation holds since for quasiprimary states v_i, the Möbius transformation $x \mapsto y = \epsilon/x$ induces

$$Y(v_i, x_i) \mapsto \epsilon^{L(0)} Y^\dagger(v_i, x_i)\epsilon^{-L(0)} = (-\frac{\epsilon}{x_i^2})^{wt(v_i)} Y(v_i, y_i).$$

Thus we find

Proposition 5.1 *For quasiprimary states v_i with the sewing scheme (80), we have*

$$\mathcal{F}_V^{(0)}(v_1, \ldots, v_n) =$$
$$\sum_{r\geq 0} \epsilon^r \sum_{u\in V_r} \mathrm{Res}_{z_1=0} z_1^{-1} Z_V^{(0)}(v_1, x_1; \ldots v_k, x_k; u, z_1)$$
$$\mathrm{Res}_{z_2=0} z_2^{-1} Z_V^{(0)}(v_n, y_n; \ldots v_{k+1}, y_{k+1}; \bar{u}, z_2) \prod_{1\leq i\leq k} (dx_i)^{wt(v_i)} \prod_{k+1\leq i\leq n} (dy_i)^{wt(v_i)}. \quad \square$$

5.2 Genus one

5.2.1 Self-sewing the Riemann sphere

We now consider genus one n-point functions defined in terms of self-sewings of a Riemann sphere. We first consider the case where punctures are located at the origin and the point at infinity. Choose local coordinates $z_1 = z$ in the neighborhood of the origin and $z_2 = 1/z'$ for z' in the neighborhood of the point at infinity. For $a = 1, 2$ and $r_a > 0$, identify the annular regions $|q|r_{\bar{a}}^{-1} \leq |z_a| \leq r_a$ for complex q satisfying $|q| \leq r_1 r_2$ via the sewing relation $z_1 z_2 = q$ i.e. $z = qz'$. Then it is straightforward to show that the annuli do not intersect for $|q| < 1$, and that $q = \exp(2\pi i\tau)$ where τ is the torus modular parameter (e.g. [MT2], Proposition 8).

We define the genus one partition function by

$$Z_V^{(1)}(q) = Z_V^{(1)}(\tau) =$$
$$\sum_{n\geq 0} q^n \sum_{u\in V_n} \mathrm{Res}_{z_2=0} z_2^{-1} \mathrm{Res}_{z_1=0} z_1^{-1} \langle \mathbf{1}, Y^\dagger(u, z_2) Y(\bar{u}, z_1)\mathbf{1}\rangle, \qquad (82)$$

where the inner sum is taken over any basis for V_n. Note that the genus zero partition function is recovered in the Riemann sphere degeneration limit $q \to 0$ where $Z_V^{(1)}(\tau) \to Z_V^{(0)} = 1$. From (70) and (77) it follows that

$$Z_V^{(1)}(\tau) = \sum_{n\geq 0} \dim V_n q^n = Tr_V(q^{L(0)}), \qquad (83)$$

the standard graded trace definition (excluding, for the present, the extra $q^{-c/24}$ factor). The genus one n-point function is similarly given by

$$\sum_{r\geq 0} q^r \sum_{u\in V_r} \mathrm{Res}_{z_2=0} z_2^{-1} \mathrm{Res}_{z_1=0} z_1^{-1} \langle \mathbf{1}, Y^\dagger(u, z_2) Y(v_1, x_1) \ldots Y(v_n, x_n) Y(\bar{u}, z_1)\mathbf{1}\rangle$$
$$= Tr_V(Y(v_1, x_1) \ldots Y(v_n, x_n) q^{L(0)}).$$

It is natural to consider the conformal map $x = q_z \equiv \exp(z)$ in order to describe the elliptic properties of the n-point function [Z1]. Since from (63), for a primary state v, $Y(v, w) \to Y(q_z^{L(0)} v, q_z)$ under this conformal map, we are led to the following definition of the genus one n-point function (op. cite.):

$$Z_V^{(1)}(v_1, z_1; \ldots v_n, z_n; \tau) =$$
$$Tr_V(Y(q_{z_1}^{L(0)} v_1, q_{z_1}) \ldots Y(q_{z_n}^{L(0)} v_n, q_{z_n}) q^{L(0)}). \tag{84}$$

Note again that $Z_V^{(0)}(v_1, z_1; \ldots v_n, z_n)$ is recovered in the degeneration limit $q \to 0$. For homogeneous primary states v_i of weight $wt(v_i)$, $Z_V^{(1)}$ parameterizes a global meromorphic differential form on the torus

$$\mathcal{F}_V^{(1)}(v_1, \ldots v_n; \tau) = Z_V^{(1)}(v_1, z_1; \ldots v_n, z_n; \tau) \prod_{1\leq i\leq n} (dz_i)^{wt(v_i)}.$$

Zhu introduced ([Z1]) a second VOA $(V, Y[,], \mathbf{1}, \tilde{\omega})$ which is isomorphic to $(V, Y(,), \mathbf{1}, \omega)$. It has vertex operators

$$Y[v, z] = \sum_{n\in\mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1), \tag{85}$$

and conformal vector $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. Let

$$Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n] z^{-n-2}, \tag{86}$$

and write $wt[v] = k$ if $L[0]v = kv$, $V_{[k]} = \{v \in V | wt[v] = k\}$. Only primary vectors are homogeneous with respect to both $L(0)$ and $L[0]$, in which case $wt(v) = wt[v]$. Similarly, we define the square bracket LiZ metric $\langle\ ,\ \rangle_{\mathrm{sq}}$ which is invariant with respect to the square bracket adjoint.

An explicit description of the n-point functions for Heisenberg and lattice VOAs is given in [MT1], using a Fock basis of $L[0]$-homogeneous states. We will make extensive use of the 1-point and 2-point functions for these VOAs below. We denote any 1-point function by

$$Z_V^{(1)}(v,\tau)=Z_V^{(1)}(v,z;\tau)=Tr_V(o(v)q^{L(0)}). \tag{87}$$

$Z^{(1)}(v,\tau)$ is independent of z since only the zero mode $o(v)$ contributes to the trace. Any n-point function can be expressed in terms of 1-point functions ([MT1], Lemma 3.1) as follows:

$$\begin{aligned}
&Z_V^{(1)}(v_1,z_1;\dots v_n,z_n;\tau)\\
=\;&Z_V^{(1)}(Y[v_1,z_1]\dots Y[v_{n-1},z_{n-1}]Y[v_n,z_n]\mathbf{1},\tau) &(88)\\
=\;&Z_V^{(1)}(Y[v_1,z_{1n}]\dots Y[v_{n-1},z_{n-1n}]v_n,\tau), &(89)
\end{aligned}$$

where $z_{in}=z_i-z_n$. Thus $Z_V^{(1)}(v_1,z_1;v_2,z_2;\tau)$ is a function of z_{12} only, which we denote by

$$\begin{aligned}
Z_V^{(1)}(v_1,v_2,z_{12},\tau) &= Z_V^{(1)}(v_1,z_1;v_2,z_2;\tau)\\
&= Tr_V(o(Y[v_1,z_{12}]v_2)q^{L(0)}). &(90)
\end{aligned}$$

We may consider a trivial sewing of a torus with local coordinate z_1 to a Riemann sphere with local coordinate z_2 by identifying the annuli $r_a\geq |z_a|\geq |\epsilon|r_{\bar{a}}^{-1}$ via the sewing relation $z_1z_2=\epsilon$. Consider $Z_V^{(1)}(v_1,x_1;\dots v_n,x_n)$ for quasi-primary v_i of $L[0]$ weight $wt[v_i]$, with $r_1\geq |x_i|\geq |\epsilon|r_2^{-1}$, and let $y_i=\epsilon/x_i$. Using (88), and employing the square bracket version of (81) with square bracket LiZ metric $\langle\ ,\rangle_{\mathrm{sq}}$, we have

$$\begin{gathered}
Z_V^{(1)}(v_1,x_1;\dots v_n,x_n;\tau)=\\
\sum_{r\geq 0}\sum_{u\in V_{[r]}} Z_V^{(1)}(Y[v_1,x_1]\dots Y[v_k,x_k]u;\tau)\langle \bar{u},Y[v_{k+1},x_{k+1}]\dots Y[v_n,x_n]\mathbf{1}\rangle_{\mathrm{sq}},
\end{gathered}$$

where the inner sum is taken over any basis $\{u\}$ of $V_{[r]}$, and $\{\bar{u}\}$ is the dual basis with respect to $\langle\ ,\rangle_{\mathrm{sq}}$. Now

$$Z_V^{(1)}(Y[v_1,x_1]\dots Y[v_k,x_k]u;\tau)=\mathrm{Res}_{z_1=0}z_1^{-1}Z_V^{(1)}(v_1,x_1;\dots v_k,x_k;u,z_1;\tau).$$

Using the isomorphism between the round and square bracket formalisms, we find as before that

$$\begin{aligned}
&\langle \bar{u},Y[v_{k+1},x_{k+1}]\dots Y[v_n,x_n]\mathbf{1}\rangle_{\mathrm{sq}}\\
=\;&\epsilon^r\mathrm{Res}_{z_2=0}z_2^{-1}Z_V^{(0)}(v_n,y_n;\dots v_{k+1},y_{k+1};\bar{u},z_2)\prod_{k+1\leq i\leq n}(-\frac{\epsilon}{x_i^2})^{wt[v_i]}.
\end{aligned}$$

We thus obtain a natural analogue of Proposition 5.1:

Proposition 5.2 *For quasiprimary states v_i with the above sewing scheme, we have*

$$\mathcal{F}_V^{(1)}(v_1, \ldots, v_n; \tau) =$$

$$\sum_{r\geq 0} \epsilon^r \sum_{u \in V_{[r]}} \mathrm{Res}_{z_1=0} z_1^{-1} Z_V^{(1)}(v_1, x_1; \ldots v_k, x_k; u, z_1; \tau).$$

$$\mathrm{Res}_{z_2=0} z_2^{-1} Z_V^{(0)}(v_n, y_n; \ldots v_{k+1}, y_{k+1}; \bar{u}, z_2). \prod_{1\leq i\leq k} (dx_i)^{wt[v_i]} \prod_{k+1\leq i\leq n} (dy_i)^{wt[v_i]},$$

where the inner sum is taken over any basis $\{u\}$ for $V_{[r]}$ and $\{\bar{u}\}$ is the dual basis with respect to $\langle\ ,\rangle_{\mathrm{sq}}$. □

5.2.2 An alternative self-sewing of the Riemann sphere and the Catalan series

We now consider an alternative construction of a torus by self-sewing a Riemann sphere with punctures located at the origin and an arbitrary point w. We will show that the resulting partition function is again (83). Choose local coordinates z_1 in the neighborhood of the origin and $z_2 = z - w$ for z in the neighborhood of w. For a complex sewing parameter ρ, identify the annuli $|\rho| r_{\bar{a}}^{-1} \leq |z_a| \leq r_a$ for $a = 1, 2$ and $|\rho| \leq r_1 r_2$ via the sewing relation

$$z_1 z_2 = \rho. \tag{91}$$

With χ as in (46), the annuli do not intersect provided $|\chi| < \frac{1}{4}$ ([MT2]) and the torus modular parameter is (Proposition 9, op.cite.)

$$q = f(\chi), \tag{92}$$

where $f(\chi)$ satisfies $f = \chi(1 + f)^2$. Thus $f(\chi)$ is the Catalan series (48) familiar from combinatorics (cf. [St]). We note the following identity (which can be proved by induction)

Lemma 5.3 *$f(\chi)$ satisfies*

$$f(\chi)^m = \sum_{n\geq m} \frac{m}{n} \binom{2n}{n+m} \chi^n. \quad \square$$

We now define the genus one partition function in the sewing scheme (91) by

$$
\begin{aligned}
&Z^{(1)}_{V,\rho}(\rho, w) = \\
&\sum_{n\geq 0}\rho^n \sum_{u\in V_n} \mathrm{Res}_{z_2=0} z_2^{-1} \mathrm{Res}_{z_1=0} z_1^{-1} \langle \mathbf{1}, Y(u, w+z_2)Y(\bar{u}, z_1)\mathbf{1}\rangle, \quad (93)
\end{aligned}
$$

where the ρ subscript refers to the sewing scheme we are currently using. In fact, this partition function is equivalent to $Z^{(1)}_V(q)$:

Theorem 5.4 *In the sewing scheme (91), we have*

$$
Z^{(1)}_{V,\rho}(\rho, w) = Z^{(1)}_V(q). \tag{94}
$$

where $q = f(\chi)$ is given by (92).

Proof. The summand in (93) is

$$
\begin{aligned}
\langle \mathbf{1}, Y(u, w)\bar{u}\rangle &= \langle Y^{\dagger}(u, w)\mathbf{1}, \bar{u}\rangle \\
&= (-w^{-2})^n \langle Y(e^{wL(1)}u, w^{-1})\mathbf{1}, \bar{u}\rangle \\
&= (-w^{-2})^n \langle e^{w^{-1}L(-1)} e^{wL(1)} u, \bar{u}\rangle,
\end{aligned}
$$

where we have used (61) and also $Y(v, z)\mathbf{1} = \exp(zL(-1))v$. (See [Ka] or [MN] for the latter equality.) Hence we find that

$$
\begin{aligned}
Z^{(1)}_{V,\rho}(\rho, w) &= \sum_{n\geq 0} \left(-\frac{\rho}{w^2}\right)^n \sum_{u\in V_n} \langle e^{w^{-1}L(-1)} e^{wL(1)} u, \bar{u}\rangle \\
&= \sum_{n\geq 0} \chi^n Tr_{V_n}(e^{w^{-1}L(-1)} e^{wL(1)}).
\end{aligned}
$$

Expanding the exponentials yields

$$
Z^{(1)}_{V,\rho}(\rho, w) = Tr_V\Big(\sum_{r\geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} \chi^{L(0)}\Big), \tag{95}
$$

an expression which depends only on χ.

In order to compute (95) we consider the quasi-primary decomposition of V. Let $Q_m = \{v \in V_m | L(1)v = 0\}$ denote the space of quasiprimary states

of weight $m \geq 1$. Then $\dim Q_m = p_m - p_{m-1}$ with $p_m = \dim V_m$. Consider the decomposition of V into $L(-1)$-descendents of quasi-primaries

$$V_n = \bigoplus_{m=1}^{n} L(-1)^{n-m} Q_m. \tag{96}$$

Lemma 5.5 *Let $v \in Q_m$ for $m \geq 1$. For an integer $n \geq m$,*

$$\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m} v = \binom{2n-1}{n-m} L(-1)^{n-m} v.$$

Proof. First use induction on $t \geq 0$ to show that

$$L(1)L(-1)^t v = t(2m + t - 1)L(-1)^{t-1} v.$$

Then by induction in r it follows that

$$\frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m} v = \binom{n-m}{r} \binom{n+m-1}{r} L(-1)^{n-m} v.$$

Hence

$$\begin{aligned} \sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m} v &= \sum_{r \geq 0}^{n-m} \binom{n-m}{r} \binom{n+m-1}{r} L(-1)^{n-m} v, \\ &= \binom{2n-1}{n-m} L(-1)^{n-m} v, \end{aligned}$$

where the last combinatorial identity follows from a comparison of the x^{n-m} coefficient of both sides of $(1+x)^{n-m}(1+x)^{n+m-1} = (1+x)^{2n-1}$. $\square$

Lemma 5.5 and (96) imply that for $n \geq 1$,

$$\begin{aligned} Tr_{V_n}\Big(\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2}\Big) &= \sum_{m=1}^{n} Tr_{Q_m}\Big(\sum_{r \geq 0} \frac{L(-1)^r L(1)^r}{(r!)^2} L(-1)^{n-m}\Big) \\ &= \sum_{m=1}^{n} (p_m - p_{m-1}) \binom{2n-1}{n-m}. \end{aligned}$$

The coefficient of p_m is

$$\binom{2n-1}{n-m} - \binom{2n-1}{n-m-1} = \frac{m}{n} \binom{2n}{m+n},$$

and hence

$$Tr_{V_n}(\sum_{r\geq 0}\frac{L(-1)^r L(1)^r}{(r!)^2}) = \sum_{m=1}^{n}\frac{m}{n}\binom{2n}{m+n}p_m.$$

Using Lemma 5.3, we find that

$$\begin{aligned}
Z^{(1)}_{V,\rho}(\rho,w) &= 1+\sum_{n\geq 1}\chi^n\sum_{m=1}^{n}\frac{m}{n}\binom{2n}{m+n}p_m,\\
&= 1+\sum_{m\geq 1}p_m\sum_{n\geq m}\frac{m}{n}\binom{2n}{m+n}\chi^n\\
&= 1+\sum_{m\geq 1}p_m(f(\chi))^m,
\end{aligned}$$

and Theorem 5.4 follows. $\square$

5.3 Genus two

We now discuss two separate definitions of the genus two n-point function associated respectively with the ϵ- and ρ-sewing schemes reviewed in Sections 2.3 and 2.4.

In the ϵ-sewing scheme we consider a pair of tori $\mathcal{S}_1, \mathcal{S}_2$ with modular parameters τ_1, τ_2 respectively. We sew them together via the sewing relation (22) of Section 2.3. By definition, in the limit $\epsilon \to 0$ the genus two surface degenerates to two tori. For $x_1, \ldots, x_k \in \mathcal{S}_1$ with $|x_i| \geq |\epsilon|/r_2$ and $y_{k+1}, \ldots, y_n \in \mathcal{S}_2$ with $|y_i| \geq |\epsilon|/r_1$ we define the genus two n-point function in the ϵ-formalism by

$$\begin{aligned}
&Z^{(2)}_{V,\epsilon}(v_1,x_1;\ldots v_k,x_k;v_{k+1},y_{k+1};\ldots v_n,y_n;\tau_1,\tau_2,\epsilon) =\\
&\sum_{r\geq 0}\epsilon^r\sum_{u\in V_{[r]}}\mathrm{Res}_{z_1=0}z_1^{-1}Z^{(1)}_V(v_1,x_1;\ldots v_k,x_k;u,z_1;\tau_1).\\
&\quad\mathrm{Res}_{z_2=0}z_2^{-1}Z^{(1)}_V(v_{k+1},y_{k+1};\ldots v_n,y_n;\bar{u},z_2;\tau_1),. \qquad (97)
\end{aligned}$$

As before, the inner sum is taken over any basis $\{u\}$ for $V_{[r]}$ and $\{\bar{u}\}$ is the dual basis with respect to $\langle\ ,\rangle_{\mathrm{sq}}$. This definition is motivated by Proposition 5.2. In this paper we will concentrate on the genus two partition function,

i.e. the 0-point function. (A discussion of genus two n-point functions will appear elsewhere [MT4]). In the notation of (87), this is given by

$$Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)=\sum_{n\geq 0}\epsilon^n\sum_{u\in V_{[n]}}Z^{(1)}_V(u,\tau_1)Z^{(1)}_V(\bar{u},\tau_2). \tag{98}$$

Note again that $Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)\rightarrow Z^{(1)}_V(\tau_1)Z^{(1)}_V(\tau_2)$ in the two tori degeneration limit $\epsilon\rightarrow 0$.

In the ρ-sewing scheme we self-sew a torus $\mathcal{S}$ with modular parameter τ via the sewing relation (35). For $x_1,\ldots,x_n\in\mathcal{S}$ with $|x_i|\geq|\epsilon|/r_2$ and $|x_i-w|\geq|\epsilon|/r_1$, we define the genus two n-point function in the ρ-formalism by

$$\begin{gathered}Z^{(2)}_{V,\rho}(v_1,x_1;\ldots v_n,x_n;\tau,w,\rho)=\\ \sum_{r\geq 0}\rho^r\sum_{u\in V_{[r]}}\mathrm{Res}_{z_1=0}z_1^{-1}\mathrm{Res}_{z_2=0}z_2^{-1}Z^{(1)}_V(\bar{u},w+z_2;v_1,x_1;\ldots v_n,x_n;u,z_1;\tau).\end{gathered} \tag{99}$$

This definition is motivated by Proposition 5.2 and (93). With the notation (90), the genus two partition function is then

$$Z^{(2)}_{V,\rho}(\tau,w,\rho)=\sum_{n\geq 0}\rho^n\sum_{u\in V_{[n]}}Z^{(1)}_V(\bar{u},u,w,\tau). \tag{100}$$

We next consider $Z^{(2)}_{V,\rho}(\tau,w,\rho)$ in the two-tori degeneration limit $w,\rho\rightarrow 0$ for fixed $|\chi|<\frac{1}{4}$ of (46) as reviewed in Section 2.3. We then find that $Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)$ and $Z^{(2)}_{V,\rho}(\tau,w,\rho)$ agree in this limit:

Theorem 5.6 *For fixed $|\chi|<\frac{1}{4}$ we have*

$$\lim_{w,\rho\rightarrow 0}Z^{(2)}_{V,\rho}(\tau,w,\rho)=Z^{(1)}_V(q)Z^{(1)}_V(f(\chi)),$$

where $f(\chi)$ is the Catalan series (48).

Proof. Using (90), the inner sum in (100) is

$$Z^{(1)}_V(\bar{u},u,w,\tau)=Tr_V(o(Y[\bar{u},w]u)q^{L(0)}).$$

Using the square bracket version of (81), we have

$$Y[\bar{u}, w]u = \sum_{m\geq 0} \sum_{v\in V_{[m]}} \langle \bar{v}, Y[\bar{u}, w]u\rangle_{\rm sq} v.$$

Similarly to the first part of Theorem 5.4, we also have

$$\begin{aligned}
\langle \bar{v}, Y[\bar{u}, w]u\rangle_{\rm sq} &= (-w^{-2})^n \langle Y[e^{wL[1]}\bar{u}, w^{-1}]\bar{v}, u\rangle_{\rm sq} \\
&= (-w^{-2})^n \langle e^{w^{-1}L[-1]} Y[\bar{v}, -w^{-1}] e^{wL[1]}\bar{u}, u\rangle_{\rm sq} \\
&= (-w^{-2})^n \langle E[\bar{v}, w]\bar{u}, u\rangle_{\rm sq},
\end{aligned}$$

where

$$E[\bar{v}, w] = \exp(w^{-1}L[-1]) Y[\bar{v}, -w^{-1}] \exp(wL[1]).$$

Hence we find

$$\begin{aligned}
Z^{(2)}_{V,\rho}(\tau, w, \rho) &= \sum_{m\geq 0} \sum_{v\in V_{[m]}} \sum_{n\geq 0} \chi^n \sum_{u\in V[n]} \langle E[\bar{v}, w]\bar{u}, u\rangle Z^{(1)}_V(v, q) \\
&= \sum_{m\geq 0} \sum_{v\in V_{[m]}} Tr_V(E[\bar{v}, w]\chi^{L[0]}) Z^{(1)}_V(v, q).
\end{aligned}$$

Now consider

$$Tr_V(E[\bar{v}, w]\chi^{L[0]}) =$$
$$w^m \sum_{r,s\geq 0} (-1)^{r+m} \frac{1}{r!s!} Tr_V(L[-1]^r \bar{v}[r-s-m-1] L[1]^s \chi^{L[0]}).$$

The leading term in w thus arises from $\bar{v} = \mathbf{1}$ of weight $m = 0$, and is equal to

$$tr_V(E[\mathbf{1}, w]\chi^{L[0]}) = Z^{(1)}_V(f(\chi)).$$

This follows from (95) and the isomorphism between the original and square bracket formalisms. Taking $w \to 0$ for fixed χ, we find $Z^{(2)}_{V,\rho}(\tau, w, \rho) \to Z^{(1)}_V(q) Z^{(1)}_V(f(\chi))$. $\square$

6 Genus two partition function for free bosons in the ϵ-formalism

6.1 The genus two partition function $Z^{(2)}_M(\tau_1, \tau_2, \epsilon)$

We begin by recalling the general definition (98) of the genus two partition function associated to a VOA V in the ϵ formalism:

$$Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)=Z^{(2)}_{V}(\tau_1,\tau_2,\epsilon)=\sum_{n\geq 0}\epsilon^n\sum_{u\in V_{[n]}}Z^{(1)}_V(u,\tau_1)Z^{(1)}_V(\bar{u},\tau_2). \tag{101}$$

The main ingredients in (101) are as follows: $\{u\}$ is a basis for $V_{[n]}, n\geq 0$; $\{\bar{u}\}$ is the dual basis with respect to the square-bracket Li-Z metric $\langle\ ,\rangle_{\mathrm{sq}}$, and $Z^{(1)}_V(u,\tau)$ is the genus one graded trace of a state u with respect to $(V,Y(\ ,))$. In this section we obtain closed formulas for $Z^{(2)}_V(\tau_1,\tau_2,\epsilon)$ in terms of an infinite determinant and also as an infinite product. We also discuss its modular properties.

The partition function (101) is independent of the choice of basis, in particular if we can choose a diagonal basis $\{u\}$ then $\bar{u}=u/\langle u,u\rangle_{\mathrm{sq}}$ and we obtain

$$Z^{(2)}_{V}(\tau_1,\tau_2,\epsilon)=\sum_{n\geq 0}\epsilon^n\sum_{u\in V_{[n]}}\frac{Z^{(1)}_V(u,\tau_1)Z^{(1)}_V(u,\tau_2)}{\langle u,u\rangle_{\mathrm{sq}}}. \tag{102}$$

Let us consider the case of the rank one Heisenberg (free boson) VOA M. Recalling the definition (25) we wish to establish the following closed formula:

Theorem 6.1 *Let M be the vertex operator algebra of one free boson. Then*

$$Z^{(2)}_{M}(\tau_1,\tau_2,\epsilon)=Z^{(1)}_M(\tau_1)Z^{(1)}_M(\tau_2)(\det(1-A_1A_2))^{-1/2}, \tag{103}$$

where A_1 and A_2 are as in (24) and $Z^{(1)}_M(\tau)=q^{1/24}/\eta(\tau)$.

Remark 6.2 *From Remark 4.2 it follows that (101) is multiplicative over tensor products of vertex operator algebras. Thus the genus two partition function for l free bosons M^l is just the l^{th} power of (103).*

Proof of Theorem. In the following we use the notation and results of [MT1] concerning the 1-point functions $Z^{(1)}_M(u,\tau)$, noting the absence of an overall $q^{-c/24}$ factor in (83) and (87). Thus $Z^{(1)}_M(\tau)=q^{1/24}/\eta(\tau)$. We take as our diagonal basis of $(V,Y[,])$ the standard Fock vectors (in the square bracket formulation)

$$v=a[-1]^{e_1}...a[-p]^{e_p}\mathbf{1}. \tag{104}$$

Of course, these Fock vectors correspond in a natural 1-1 manner with unrestricted partitions, the state v (104) corresponding to a partition $\lambda = \{1^{e_1}...p^{e_p}\}$ of $n = \sum_{1\leq i\leq p} ie_i$. We sometimes write $v = v(\lambda)$ to indicate this correspondence. As discussed at length in [MT1], the partition λ may be thought of as a labelled set $\Phi = \Phi_\lambda$ with e_i elements labelled i. One of the main results of [MT1] (loc.cit. Corollary 1 and eqn.(53)) is then that

$$Z_M^{(1)}(v(\lambda),\tau) = Z_M^{(1)}(\tau) \sum_{\phi\in F(\Phi_\lambda)} \Gamma(\phi), \tag{105}$$

with

$$\Gamma(\phi,\tau) = \Gamma(\phi) = \prod_{(r,s)} C(r,s,\tau), \tag{106}$$

for C of (15), where ϕ ranges over the elements of $F(\Phi_\lambda)$ (the fixed-point-free involutions in $\Sigma(\Phi_\lambda)$) and (r,s) ranges over the orbits of ϕ on Φ_λ.

With this notation, and using the Fock basis (104) as well as (75), (102) reads

$$Z_M^{(2)}(\tau_1,\tau_2,\epsilon) = Z_M^{(1)}(\tau_1)Z_M^{(1)}(\tau_1) \sum_{\lambda=\{i^{e_i}\}} \frac{E(\lambda)}{\prod_i(-i)^{e_i}e_i!}\epsilon^{\sum ie_i}, \tag{107}$$

where λ ranges over all unrestricted partitions and where we have set

$$E(\lambda) = \sum_{\phi,\psi\in F(\Phi_\lambda)} \Gamma_1(\phi)\Gamma_2(\psi), \tag{108}$$

$$\Gamma_i(\phi) = \Gamma(\phi,\tau_i). \tag{109}$$

We now analyze the nature of the expression $E(\lambda)$ more closely. This will lead us to the connection between $Z^{(2)}(\tau_1,\tau_2,\epsilon)$ and the chequered cycles discussed in Section 3.1. The idea is to use the technique employed in the proof of Proposition 3.10 of [MT1]. If we fix for a moment a partition λ then a pair of fixed-point-free involutions ϕ,ψ correspond (loc.cit.) to a pair of complete matchings μ_ϕ,μ_ψ on the labelled set Φ_λ which we may represent pictorially as

$$\begin{array}{ccccc}
\overset{r_1}{\bullet} & \xrightarrow{1} & \overset{s_1}{\bullet} & \xrightarrow{2} & \overset{t_1}{\bullet} \\
\overset{r_2}{\bullet} & \xrightarrow{1} & \overset{s_2}{\bullet} & \xrightarrow{2} & \overset{t_2}{\bullet} \\
\vdots & & \vdots & \vdots & \vdots \\
\overset{r_b}{\bullet} & \xrightarrow{1} & \overset{s_b}{\bullet} & \xrightarrow{2} & \overset{t_b}{\bullet}
\end{array}$$

Fig. 5 Two complete matchings

Here, μ_ϕ is the matching with edges labelled 1, μ_ψ the matching with edges labelled 2, and where we have denoted the (labelled) elements of Φ_λ by $\{r_1, s_1, ..., r_b, s_b\} = \{s_1, t_1, ..., s_b, t_b\}$. From this data we may create a chequered cycle in a natural way: starting with some node of Φ_λ, apply the involutions ϕ, ψ successively and repeatedly until the initial node is reached, using the complete matchings to generate a chequered cycle. The resulting chequered cycle corresponds to an orbit of $\langle\psi\phi\rangle$ considered as a cyclic subgroup of $\Sigma(\Phi_\lambda)$. Repeat this process for each such orbit to obtain a *chequered diagram* D consisting of the union of the chequered cycles corresponding to all of the orbits of $\langle\psi\phi\rangle$ on Φ_λ. To illustrate, for the partition $\lambda = \{1^2.2.3^2.5\}$ with matchings $\mu_\phi = (13)(15)(23)$ and $\mu_\psi = (11)(35)(23)$, the corresponding chequered diagram is

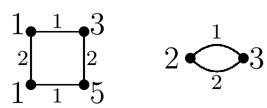

Fig. 6 Chequered Diagram

As usual, two chequered diagrams are isomorphic if there is a bijection on the nodes which preserves edges and labels of nodes and edges. If $\lambda = \{1^{e_1}...p^{e_p}\}$ then $\Sigma(\Phi_\lambda)$ acts on the chequered diagrams which have Φ_λ as underlying set of labelled nodes. The *label subgroup* Λ, consisting of the elements of $\Sigma(\Phi_\lambda)$ which preserves node labels, is isomorphic to $\Sigma_{e_1} \times \ldots \times \Sigma_{e_p}$. It induces all isomorphisms among these chequered diagrams. Of course $|\Lambda| = \prod_{1 \leq i \leq p} e_i!$. We have almost established the first step in the proof of Theorem 6.1, namely

Proposition 6.3 *We have*

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1) Z_M^{(1)}(\tau_1) \sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|}, \tag{110}$$

where D ranges over isomorphism classes of chequered configurations and

$$\gamma(D) = \frac{E(\lambda)}{\prod_i (-i)^{e_i}} \epsilon^{\sum i e_i}. \tag{111}$$

Proposition 6.3 follows from what we have said together with (107). It is only necessary to point out that because the label subgroup induces all isomorphisms of chequered diagrams, when we sum over isomorphism classes of such diagrams in (107) the term $\prod_i e_i!$ must be replaced by $|\mathrm{Aut}(D)|$. $\square$

Recalling the weights (53), we define

$$\omega(D) = \Pi_E \omega(E),$$

where the product is taken over the edges E of D and $\omega(E)$ is as in (57).

Lemma 6.4 *For all D we have*

$$\omega(D) = \gamma(D). \tag{112}$$

Proof. Let D be determined by a partition $\lambda = \{1^{e_1} ... p^{e_p}\}$ and a pair of involutions $\phi, \psi \in F(\Phi_\lambda)$, and let $(a,b), (r,s)$ range over the orbits of ϕ resp. ψ on Φ_λ. Then we find

$$\begin{aligned}
\frac{E(\lambda)}{\prod_i (-i)^{e_i}} \epsilon^{\sum i e_i} &= \frac{\prod_{(a,b)} C(a,b,\tau_1) \prod_{(r,s)} C(r,s,\tau_2)}{\prod_i (-i)^{e_i}} \epsilon^{\sum i e_i} \\
&= (-1)^{\sum e_i} \prod_{(ab)} \frac{\epsilon^{(a+b)/2}}{\sqrt{ab}} C(a,b,\tau_1) \prod_{(rs)} \frac{\epsilon^{(r+s)/2}}{\sqrt{rs}} C(r,s,\tau_1) \\
&= (-1)^{\sum e_i} \prod_{(ab)} A_1(a,b) \prod_{(rs)} A_2(r,s) \\
&= (-1)^{\sum e_i} \omega(D).
\end{aligned}$$

For $\sum e_i$ even this is (112), whereas if $\sum e_i$ is odd then $\omega(D)$ vanishes since some Eisenstein series $E_{a+b}(\tau)$ vanishes. $\square$

We may represent a chequered diagram formally as a product

$$D = \prod_i L_i^{m_i} \tag{113}$$

in case D is the disjoint union of unoriented chequered cycles L_i with multiplicity m_i. Then $\mathrm{Aut}(D)$ is isomorphic to the direct product of the groups $\mathrm{Aut}(L_i^{m_i})$, and

$$|\mathrm{Aut}(D)| = \prod_i |\mathrm{Aut}(L_i^{m_i})| m_i!$$

Noting that the expression $\omega(D)$ is multiplicative over disjoint unions of diagrams, we calculate

$$\begin{aligned}
\sum_D \frac{\omega(D)}{|\mathrm{Aut}(D)|} &= \prod_L \sum_{k\geq 0} \frac{\omega(L^k)}{|\mathrm{Aut}(L^k)|} \\
&= \prod_L \sum_{k\geq 0} \frac{\omega(L)^k}{|\mathrm{Aut}(L)|^k k!} \\
&= \prod_L \exp\left(\frac{\omega(L)}{|\mathrm{Aut}(L)|}\right) \\
&= \exp\left(\sum_L \frac{\omega(L)}{|\mathrm{Aut}(L)|}\right),
\end{aligned}$$

where L ranges over isomorphism classes of unoriented chequered cycles. Now $\mathrm{Aut}(L)$ is either a dihedral group of order $2r$ or a cyclic group of order r for some $r \geq 1$, depending on whether L admits a reflection symmetry or not. If we now *orient* our cycles, say in a clockwise direction, then we can replace the previous sum over L by a sum over the set of (isomorphism classes of) *oriented* chequered cycles $\mathcal{O}$ to obtain

$$\sum_D \frac{\omega(D)}{|\mathrm{Aut}(D)|} = \exp\left(\frac{1}{2}\sum_{M\in\mathcal{O}} \frac{\omega(M)}{|\mathrm{Aut}(M)|}\right). \tag{114}$$

Let $\mathcal{O}_{2n} \subset \mathcal{O}$ denoted the set of oriented chequered cycles with $2n$ nodes. Then we have

Lemma 6.5

$$Tr((A_1A_2)^n) = \sum_{M\in\mathcal{O}_{2n}} \frac{n}{|\mathrm{Aut}(M)|}\omega(M). \tag{115}$$

Proof. The contribution $A_1(i_1,i_2)A_2(i_2,i_3)\ldots A_2(i_{2n},i_1)$ to the left-hand-side of (115) is equal to the weight $\omega(M)$ for some $M \in \mathcal{O}_{2n}$ with vertices $i_1, i_2, \ldots i_{2n}$. Let $\sigma = \begin{pmatrix} i_1 & \ldots & i_k & \ldots & i_{2n} \\ i_3 & \ldots & i_{k+2} & \ldots & i_2 \end{pmatrix}$ denote the order n permutation of the indices which generates rotations of M. Then $\mathrm{Aut}(M) = \langle\sigma^m\rangle$ for some $m = n/|\mathrm{Aut}(M)|$. Now sum over all i_k to compute $Tr((A_1A_2)^n)$,

noting that for inequivalent M the weight $\omega(M)$ occurs with multiplicity m. The Lemma follows. $\square$

We may now complete the proof of Theorem 6.1. From (114) and (115) we obtain

$$\begin{aligned}
\sum_D \frac{\omega(D)}{|\mathrm{Aut}(D)|} &= \exp\left(\frac{1}{2}tr(\sum_n \frac{1}{n}(A_1A_2)^n)\right) \\
&= \exp(-\frac{1}{2}tr(\log(1-A_1A_2))) \\
&= \det(\exp(-\frac{1}{2}(\log(1-A_1A_2)))) \\
&= (\det(1-A_1A_2))^{-1/2}. \quad \square
\end{aligned}$$

We may also obtain a product formula for $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ as follows. Recalling the notation (51), for each oriented chequered cycle M, $\mathrm{Aut}(M)$ is a cyclic group of order r for some $r \geq 1$. Furthermore it is evident that there is a rotationless chequered cycle N with $\omega(M) = \omega(N)^r$. Indeed, N may be obtained by taking a suitable consecutive sequence of n/r nodes of M, where n is the total number of nodes of M. We thus see that

$$\begin{aligned}
\sum_{M\in\mathcal{O}} \frac{\omega(M)}{|\mathrm{Aut}(M)|} &= \sum_{N\in\mathcal{R}}\sum_{r\geq 1}\frac{\omega(N)^r}{r} \\
&= -\sum_{N\in\mathcal{R}}\log(1-\omega(N)).
\end{aligned}$$

Then (114) implies

$$\det(1-A_1A_2) = \prod_{N\in\mathcal{R}}(1-\omega(N)), \tag{116}$$

and thus we obtain

Theorem 6.6 *Let M be the vertex operator algebra of one free boson. Then*

$$Z_M^{(2)}(\tau_1,\tau_2,\epsilon) = \frac{Z_M^{(1)}(\tau_1)Z_M^{(1)}(\tau_2)}{\prod_{N\in\mathcal{R}}(1-\omega(N))^{1/2}}. \tag{117}$$

6.2 Holomorphic and modular-invariance properties

In Section 2.2 we reviewed the genus two ϵ-sewing formalism and introduced the domain $\mathcal{D}^{\epsilon}$ parameterizing the genus two surface. An immediate consequence of Theorem 6.1 and Theorem 2.1b) is the following:

Theorem 6.7 *$Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ is holomorphic on the domain $\mathcal{D}^{\epsilon}$.* $\square$

We next consider the automorphic properties of the genus two partition function with respect to the group G reviewed in Section 2.2. For a vertex operator algebra of central charge c at genus one, we usually consider the *modular partition function*

$$Z_{V,\mathrm{mod}}^{(1)}(\tau) = q^{-c/24} Z_V^{(1)}(\tau),$$

because of its enhanced $SL(2,\mathbb{Z})$-invariance properties. In particular, for *two* free bosons this is

$$Z_{M^2,\mathrm{mod}}^{(1)}(\tau) = \frac{1}{\eta(\tau)^2}. \tag{118}$$

Let[3] χ be the character of $SL(2,\mathbb{Z})$ defined by its action on $\eta(\tau)^{-2}$, i.e.

$$\eta(\gamma\tau)^{-2} = \chi(\gamma)\eta(\tau)^{-2}(c\tau + d)^{-1}, \tag{119}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Recall (e.g. [Se]) that $\chi(\gamma)$ is a twelfth root of unity. For a function $f(\tau)$ on $\mathbb{H}_1$, $k \in \mathbb{Z}$ and $\gamma \in SL(2,\mathbb{Z})$, we define

$$f(\tau)|_k\gamma = f(\gamma\tau)\ (c\tau + d)^{-k}, \tag{120}$$

so that

$$Z_{M^2,\mathrm{mod}}^{(1)}(\tau)|_{-1}\gamma = \chi(\gamma) Z_{M^2,\mathrm{mod}}^{(1)}(\tau). \tag{121}$$

Similarly, at genus two we define the *genus two modular partition function* for two free bosons

$$\begin{aligned} Z_{M^2,\mathrm{mod}}^{(2)}(\tau_1, \tau_2, \epsilon) &= Z_{M^2}^{(2)}(\tau_1, \tau_2, \epsilon) q_1^{-1/12} q_2^{-1/12} \\ &= \frac{1}{\eta(\tau_1)^2 \eta(\tau_2)^2 \det(1 - A_1 A_2)}. \end{aligned} \tag{122}$$

[3] There should be no confusion between the character χ introduced here and the variable χ used in (46)

Analogously to (120), we define

$$f(\tau_1,\tau_2,\epsilon)|_k\gamma = f(\gamma(\tau_1,\tau_2,\epsilon))\det(C\Omega+D)^{-k}. \tag{123}$$

Here, the action of γ on the right-hand-side is as in (31). We have abused notation by adopting the following conventions in (123), which we continue to use below:

$$\Omega = F^{\epsilon}(\tau_1,\tau_2,\epsilon),\ \gamma = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right) \in Sp(4,\mathbb{Z}) \tag{124}$$

where F^{ϵ} is as in Theorem 2.2, and γ is identified with an element of $Sp(4,\mathbb{Z})$ via (32)-(33). Note that (123) defines a right action of G on functions $f(\tau_1,\tau_2,\epsilon)$. We will establish the natural extension of (121). To describe this, introduce the character $\chi^{(2)}$ of G defined by

$$\chi^{(2)}(\gamma_1\gamma_2\beta^m) = (-1)^m\chi(\gamma_1\gamma_2), \qquad \gamma_i \in \Gamma_a, a=1,2.$$

(Γ_i is as in (32)). Thus $\chi^{(2)}$ takes values which are twelfth roots of unity, and we have

Theorem 6.8 *If $\gamma \in G$ then*

$$Z^{(2)}_{M^2,\mathrm{mod}}(\tau_1,\tau_2,\epsilon)|_{-1}\gamma = \chi^{(2)}(\gamma)Z^{(2)}_{M^2,\mathrm{mod}}(\tau_1,\tau_2,\epsilon).$$

Corollary 6.9 *Set $Z^{(2)}_{M^{24},\mathrm{mod}} = (Z^{(2)}_{M^2,\mathrm{mod}})^{12}$. Then for $\gamma \in G$ we have*

$$Z^{(2)}_{M^{24},\mathrm{mod}}(\tau_1,\tau_2,\epsilon)|_{-12}\gamma = Z^{(2)}_{M^{24},\mathrm{mod}}(\tau_1,\tau_2,\epsilon).$$

Proof. We will give two different proofs of this result. Using the convention (124), we have to show that

$$Z^{(2)}_{M^2,\mathrm{mod}}(\gamma(\tau_1,\tau_2,\epsilon))\det(C\Omega+D) = \chi^{(2)}(\gamma)Z^{(2)}_{M^2,\mathrm{mod}}(\tau_1,\tau_2,\epsilon) \tag{125}$$

for $\gamma \in G$, and it is enough to do this for a generating set of G. If $\gamma = \beta$ then the result is clear since $\det(C\Omega+D) = \chi^{(2)}(\beta) = -1$ and β exchanges τ_1 and τ_2. So we may assume that $\gamma = (\gamma_1,\gamma_2) \in \Gamma_1 \times \Gamma_2$.

Our first proof utilizes the determinant formula (103) as follows. For $\gamma_1 \in \Gamma_1$, define $A'_a(k,l,\tau_a,\epsilon) = A_a(k,l,\gamma_1\tau_a,\frac{\epsilon}{c_1\tau_1+d_1})$ following (31). We find from Section 4.4 of [MT2] that

$$\begin{aligned} 1 - A'_1A'_2 &= 1 - A_1A_2 - \kappa\Delta A_2 \\ &= (1-\kappa S).(1-A_1A_2), \end{aligned}$$

where

$$\begin{aligned}
\Delta(k,l) &= \delta_{k1}\delta_{l1}, \\
\kappa &= -\frac{\epsilon}{2\pi i}\frac{c_1}{c_1\tau_1+d_1}, \\
S(k,l) &= \delta_{k1}(A_2(1-A_1A_2)^{-1})(1,l).
\end{aligned}$$

Since $\det(1-A_1A_2)$ and $\det(1-A_1'A_2')$ are convergent on $\mathcal{D}^\epsilon$ we find

$$\det(1-A_1'A_2') = \det(1-\kappa S)\det(1-A_1A_2).$$

But

$$\begin{aligned}
\det(1-\kappa S) &= \begin{vmatrix} 1-\kappa S(1,1) & -\kappa S(1,2) & -\kappa S(1,3) & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \\
&= 1-\kappa S(1,1) \\
&= \frac{c_1\Omega_{11}+d_1}{c_1\tau_1+d_1},
\end{aligned}$$

using (28). Thus

$$\det(1-A_1'A_2') = \frac{c_1\Omega_{11}+d_1}{c_1\tau_1+d_1}.\det(1-A_1A_2),$$

which implies (125) for $\gamma_1 \in \Gamma_1$. A similar proof applies for $\gamma_2 \in \Gamma_2$.

The second proof uses Proposition 3.1 together with (58), which tell us that

$$Z^{(2)}_{M^2,\mathrm{mod}}(\tau_1,\tau_2,\epsilon) = \frac{-2\pi i\Omega_{12}}{\epsilon\eta(\tau_1)^2\eta(\tau_2)^2}\prod_{\mathcal{R}'}(1-\omega(L))^{-1}, \tag{126}$$

where $\mathcal{R}' = \mathcal{R}\setminus\mathcal{R}_{21}$. Now in general a term $\omega(L)$ will not be invariant under the action of γ. This is because of the presence of non-modular terms $A_a(1,1)$ arising from $E_2(\tau_a)$. But it is clear from (31) and the definition (15) of $C(k,l,\tau)$ together with its modular-invariance properties that if $L \in \mathcal{R}'$ then such terms are absent and $\omega(L)$ *is* invariant. So the product term in (126) is invariant under the action of γ.

Next, we see from (31) that the expression $\epsilon\eta(\tau_1)^2\eta(\tau_2)^2$ is invariant under the action of γ up to a scalar $\chi(\gamma_1)\chi(\gamma_2) = \chi^{(2)}(\gamma)$. This reduces the proof of (125) to showing that

$$(\gamma_1, \gamma_2) : \Omega_{12} \mapsto \Omega_{12} \ \det(C\Omega + D)^{-1},$$

and this is implicit in (34) upon applying Theorem 2.3. This completes the second proof of Theorem 6.8. □

Remark 6.10 *An unusual feature of the formulas in Theorem 6.8 and Corollary 6.9 is that the definition of the automorphy factor* $\det(C\Omega + D)$ *requires the map* $F^\epsilon : D^\epsilon \to \mathbb{H}_2$.

Remark 6.11 *The reason for the appearance of the factor* $-2\pi i\Omega_{12}/\epsilon$ *in (126) can be understood as follows. From Theorem 2.2 we see that* $-2\pi i\Omega_{12}/\epsilon = (1 - A_1A_2)^{-1}(1, 1)$. *However*

$$(1 - A_1A_2)^{-1}(1, 1) = \frac{c(1, 1)}{\det(1 - A_1A_2)}, \tag{127}$$

where $c(1, 1)$ *is the* $(1, 1)$ *cofactor for* $1-A_1A_2$. *Comparison with (126) shows that* $c(1, 1) = \prod_{\mathcal{R}'}(1 - \omega(L))^{-1}$ *which contains no* $A_a(1, 1)$ *terms, and hence is modular invariant.*

7 Genus two partition function for lattice theories in the ϵ-formalism

Let L be an even lattice with V_L the corresponding lattice theory vertex operator algebra. The underlying Fock space is

$$V_L = M^l \otimes C[L] = \oplus_{\alpha\in L} M^l \otimes e^\alpha, \tag{128}$$

where M^l is the corresponding Heisenberg free boson theory of rank $l = \dim L$ based on $H = C \otimes_Z L$. We follow Section 4.1 and [MT1] concerning further notation for lattice theories.

The general shape of $Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ is as in (101). Note that the modes of a state $u \otimes e^\alpha$ map $M^l \otimes e^\beta$ to $M^l \otimes e^{\alpha+\beta}$. Thus if $\alpha \neq 0$ then $Z^{(1)}_{V_L}(u \otimes e^\alpha, \tau)$

vanishes, and as a result we see that

$$
\begin{aligned}
Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon) &= \sum_{n\geq 0}\epsilon^n \sum_{u\in M^l_{[n]}} \frac{Z^{(1)}_{V_L}(u,\tau_1)Z^{(1)}_{V_L}(u,\tau_2)}{\langle u,u\rangle_{\mathrm{sq}}} \\
&= \sum_{n\geq 0}\epsilon^n \sum_{u\in M^l_{[n]}} \sum_{\alpha,\beta\in L} \frac{Z^{(1)}_{M^l\otimes e^\alpha}(u,\tau_1)Z^{(1)}_{M^l\otimes e^\beta}(u,\tau_2)}{\langle u,u\rangle_{\mathrm{sq}}}. \qquad (129)
\end{aligned}
$$

Here, as in (102), u ranges over a diagonal basis for $M^l_{[n]}$; $M^l\otimes e^\alpha$ should be viewed as a simple module for M^l. An explicit formula for $Z^{(1)}_{M^l\otimes e^\alpha}(u,\tau)$ was given in [MT2] (Corollary 3 and Theorem 1). We are going to use these results, much as in the case of free bosons carried out in Section 6, to elucidate (129). Indeed, we will establish

Theorem 7.1 *We have*

$$
Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon) = Z^{(2)}_{M^l}(\tau_1,\tau_2,\epsilon)\theta^{(2)}_L(\Omega), \qquad (130)
$$

where $\theta^{(2)}_L(\Omega)$ is the (genus two) Siegel theta function associated to L (e.g. [F])

$$
\theta^{(2)}_L(\Omega) = \sum_{\alpha,\beta\in L} \exp(\pi i((\alpha,\alpha)\Omega_{11} - 2(\alpha,\beta)\Omega_{12} + (\beta,\beta)\Omega_{22})), \qquad (131)
$$

Remark 7.2 *Note that $\theta^{(2)}_L$ is independent of the sign of the coefficient of Ω_{12} in the exponent.*

We will deal first with the case $l=1$, when $M=M^1$, and subsequently show how to modify the argument to handle a general lattice. To this end, let a be a unit vector of H and use the Fock basis vectors $v=v(\lambda)$ (cf. (104)) identified with partitions $\lambda=\{i^{e_i}\}$ as in Section 6. Recall that λ defines a labelled set Φ_λ with e_i nodes labelled i. It is useful to re-state Corollary 3 of [MT1] in the following form:

$$
Z^{(1)}_{M\otimes e^\alpha}(v,\tau) = Z^{(1)}_M(\tau)q^{(\alpha,\alpha)/2}\sum_\phi \Gamma_{\lambda,\alpha}(\phi). \qquad (132)
$$

Here, ϕ ranges over the set of involutions

$$
\mathrm{Inv}_1(\Phi_\lambda) = \{\phi\in \mathrm{Inv}(\Phi_\lambda)|\ p\in \mathrm{Fix}(\phi)\Rightarrow p \text{ has label } 1\}. \qquad (133)
$$

(In words, ϕ is an involution in the symmetric group $\Sigma(\Phi_\lambda)$ such that all fixed-points of ϕ carry the label 1. Note that this includes the fixed-point-free involutions, which were the only involutions which played a role in the case of free bosons. The main difference between free bosonic and lattice theories is the need to include additional involutions in the latter case.) Finally,

$$\Gamma_{\lambda,\alpha}(\phi,\tau) = \Gamma_{\lambda,\alpha}(\phi) = \prod_{\Xi} \Gamma(\Xi), \tag{134}$$

where Ξ ranges over the orbits (of length ≤ 2) of ϕ acting on Φ_λ and

$$\Gamma(\Xi) = \begin{cases} C(r,s,\tau), & \text{if } \Xi = \{r,s\}, \\ (a,\alpha), & \text{if } \Xi = \{1\}. \end{cases} \tag{135}$$

From (129)-(134) we get

$$\begin{gathered} Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon) = \\ Z^{(1)}_M(\tau_1) Z^{(1)}_M(\tau_2) \sum_{\alpha,\beta\in L} \sum_{\lambda=\{i^{e_i}\}} \frac{E_{\alpha,\beta}(\lambda)}{\prod_i (-i)^{e_i} e_i!} q_1^{(\alpha,\alpha)/2} q_2^{(\beta,\beta)/2} \epsilon^{\sum i e_i}, \end{gathered} \tag{136}$$

where

$$E_{\alpha,\beta}(\lambda) = \sum_{\phi,\psi\in \mathrm{Inv}_1(\Phi_\lambda)} \Gamma_{\lambda,\alpha}(\phi,\tau_1)\Gamma_{\lambda,\beta}(\psi,\tau_2). \tag{137}$$

(Compare with eqns. (107) - (108).) Now we follow the proof of Proposition 6.3 to obtain an expression analogous to (110), namely

$$\begin{gathered} Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon) = \\ Z^{(1)}_M(\tau_1) Z^{(1)}_M(\tau_2) \sum_{\alpha,\beta\in L} \sum_{D_{\alpha,\beta}} \frac{\gamma^0(D_{\alpha,\beta})}{|\mathrm{Aut}(D_{\alpha,\beta})|} q_1^{(\alpha,\alpha)/2} q_2^{(\beta,\beta)/2}, \end{gathered} \tag{138}$$

the meaning of which we now enlarge upon. Compared to (110), the chequered diagrams $D_{\alpha,\beta}$ which occur in (138) not only depend on lattice elements but are also more general than before, in that they reflect the fact that the relevant involutions may now have fixed-points. Thus $D_{\alpha,\beta}$ is the union of its (as yet unoriented) connected components which are either chequered cycles as before or else chequered necklaces as introduced in Section 3.2. Necklaces arise from orbits of the group $\langle\psi\phi\rangle$ on Φ_λ in which one of the

nodes in the orbit is a fixed-point of ϕ or ψ. In that case the orbit will generally contain two such nodes which comprise the end nodes of the necklace. Note that these end nodes necessarily carry the label 1 (cf. (133)). There is degeneracy when *both* ϕ and ψ fix the node, in which case the degenerate necklace obtains. To be quite precise, the necklaces that we are presently dealing with are not quite the same as those of Section 3.2: not only are they unoriented, but they should be conceived as having a loop (self-edge) at each end node that carries a label (a, α) or (a, β) (or the product of the two in case of degeneracy). For convenience, we call these *chequered* necklaces* and denote their isomorphism class by $\mathcal{N}^*_{\alpha,\beta}$.

Similarly to (111), the term $\gamma^0(D_{\alpha,\beta})$ in (138) is given by

$$\gamma^0(D_{\alpha,\beta}) = \frac{\prod_{\Xi_1} \Gamma(\Xi_1) \prod_{\Xi_2} \Gamma(\Xi_2)}{\prod_i (-i)^{e_i}} \epsilon^{\sum i e_i}, \tag{139}$$

where Ξ_1, Ξ_2 range over the orbits of ϕ, ψ respectively on Φ_λ. As usual the summands in (138) are multiplicative over connected components of the chequered diagram. This applies, in particular, to the chequered cycles which occur, and these are independent of the lattice elements. As a result, (138) factors as a product of two expressions, the first a sum over diagrams consisting only of chequered cycles and the second a sum over diagrams consisting only of chequered* necklaces. However, the first expression corresponds precisely to the genus two partition function for the free boson (Proposition 6.3). We thus obtain

$$\frac{Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)}{Z^{(2)}_M(\tau_1, \tau_2, \epsilon)} = \sum_{\alpha,\beta \in L} \sum_{D^N_{\alpha,\beta}} \frac{\gamma^0(D^N_{\alpha,\beta})}{|\mathrm{Aut}(D^N_{\alpha,\beta})|} q_1^{(\alpha,\alpha)/2} q_2^{(\beta,\beta)/2}, \tag{140}$$

where here $D^N_{\alpha,\beta}$ ranges over all chequered diagrams all of whose connected components are chequered* necklaces in $\mathcal{N}^*_{\alpha,\beta}$. So (at least if $l = 1$) Theorem 7.1 is reduced to establishing

Proposition 7.3 *We have*

$$\theta^{(2)}_L(\Omega) = \sum_{\alpha,\beta \in L} \sum_{D^N_{\alpha,\beta}} \frac{\gamma^0(D^N_{\alpha,\beta})}{|\mathrm{Aut}(D^N_{\alpha,\beta})|} q_1^{(\alpha,\alpha)/2} q_2^{(\beta,\beta)/2}. \tag{141}$$

We may apply the argument of (113) et. seq. to the inner sum in (141) to write it as an exponential expression

$$\exp\{\pi i((\alpha,\alpha)\tau_1+(\beta,\beta)\tau_2)+\sum_{N^*\in\mathcal{N}^*_{\alpha,\beta}}\frac{\gamma^0(N^*)}{|\mathrm{Aut}(N^*)|}\}, \tag{142}$$

where N^* ranges over all chequered* necklaces $\mathcal{N}^*_{\alpha,\beta}$.

Recall the isomorphism class $\mathcal{N}_{ab}$ of oriented chequered necklaces of type ab as displayed in Fig.4 of Section 3.2. We can similarly consider N^* a chequered necklaces* of type ab. Then $|\mathrm{Aut}(N^*)|\leq 2$ with equality if, and only if, $a=b$ and the two orientations of N^* are isomorphic. Then (142) can be written as

$$\exp\{\pi i((\alpha,\alpha)\tau_1+(\beta,\beta)\tau_2)+\frac{1}{2}\sum_{a,b\in\{1,2\}}\sum_{N^*}\gamma^0(N^*)\}. \tag{143}$$

where here N^* ranges over *oriented* chequered* necklaces of type ab.

From (135) we see that the contribution of the end nodes to $\gamma^0(N^*)$ is equal to $(a,\alpha)^2=(\alpha,\alpha)$ for a type 11 necklace, and similarly $(\beta,\beta),(\alpha,\beta)$ and (β,α) for types $22, 12$ and 21 respectively. The remaining factors of $\gamma^0(N^*)$ have product $\gamma(N)=\omega(N)$ for some $N\in\mathcal{N}_{ab}$ by Lemma 6.4. Then (143) may be re-expressed as

$$\sum_{\alpha,\beta\in L}\exp\{\frac{1}{2}(\alpha,\alpha)(2\pi i\tau_1+\epsilon\omega_{11})+\frac{1}{2}(\beta,\beta)(2\pi i\tau_2+\epsilon\omega_{22})+(\alpha,\beta)\epsilon\omega_{12}\}, \tag{144}$$

recalling $\omega_{ab}=\sum_{N\in\mathcal{N}_{ab}}\omega(N)$ and where $\omega_{12}=\omega_{21}$. Note that in passing to $\omega(N)$ the (weight of) end nodes of each necklace contribute an additional overall factor ϵ, as displayed in (144). (144) reproduces (131) on applying Proposition 3.2. This completes the proof of Theorem 7.1 in the case of a rank 1 lattice.

We now consider the general case of an even lattice L of rank l. We again start with (129), using results of [MT1] to describe $Z^{(1)}_{M^l\otimes e^\alpha}(u,\tau)$. We carry over the notation of [MT1] (especially the paragraphs prior to Theorem 1), in particular $a_1,...,a_l$ is an orthonormal base of $H=C\otimes L$ and u is as in

eqn.(75) (loc. cit.). Then the case $n = 1$ and $\alpha_1 = 0$ of Theorem 1 (loc. cit.) tells us that

$$Z^{(1)}_{M^l\otimes e^\alpha}(u,\tau) = Z^{(1)}_{M^l}(\tau)q^{(\alpha,\alpha)/2}\prod_{r=1}^{l}\sum_{\phi\in \mathrm{Inv}_1(\Phi_{\lambda^r})}\Gamma^r(\phi). \tag{145}$$

Here, Φ_{λ^r} is the labelled set corresponding to the r^{th} partition λ^r, which itself is the partition determined by the occurrences of a_r in the representation of the state u; $\Gamma^r(\phi)$ is as in (134) with the understanding that the relevant labelled set is now that determined by λ^r. As a result, we obtain an expression for $Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon)$ analogous to (136), the only difference now being that the sum over partitions λ must be replaced by a sum over l-tuples of partitions $\lambda^r, 1 \leq r \leq l$. In terms of chequered diagrams, it is much as before except that the connected components (either chequered cycles or necklaces) have an additional overall color r in the range $1 \leq r \leq l$. Then the proof of Proposition 6.3 still applies and we obtain the analogues of (138) - (143) in which we again must sum also over the l different colors of the pertinent graphs. Explicitly, (143) now reads

$$\exp\{\pi i((\alpha,\alpha)\tau_1 + (\beta,\beta)\tau_2) + \frac{1}{2}\sum_{r=1}^{l}\sum_{a,b\in\{1,2\}}\sum_{N^{r*}}\gamma^0(N^{r*})\}. \tag{146}$$

Finally, the contribution of the end nodes of N^{r*} to $\gamma^0(N^{r*})$ is $(a_r,\alpha)^2$ for type 11 necklaces, and similarly for the other types. This being the only dependence of the expression $\gamma^0(N^{r*})$ on r, it follows as before that (146) is equal to

$$\exp\{\pi i((\alpha,\alpha)\tau_1 + (\beta,\beta)\tau_2) + \frac{\epsilon}{2}\sum_{r=1}^{l}(a_r,\alpha)^2\omega_{11}$$
$$+\epsilon\sum_{r=1}^{l}(a_r,\alpha)(a_r,\beta)\omega_{12} + \frac{\epsilon}{2}\sum_{r=1}^{l}(a_r,\beta)^2\omega_{22}\},$$

that is

$$\exp\{\frac{1}{2}(\alpha,\alpha)(2\pi i\tau_1 + \epsilon\omega_{11}) + \frac{1}{2}(\beta,\beta)(2\pi i\tau_2 + \epsilon\omega_{22}) + \epsilon(\alpha,\beta)\omega_{12}\}.$$

The remaining argument is now as before, and completes the proof of Theorem 7.1. $\square$

We complete this Section with a brief discussion of the automorphic properties of $Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon)$. There is more that one can say here, but a fuller discussion must wait for another time. The function $\theta^{(2)}_L(\Omega)$ is a Siegel modular form of weight $l/2$ ([F]), in particular it is holomorphic on the Siegel upper half-space $\mathbb{H}_2$. From Theorems 2.2, 6.7 and 7.1, we deduce

Theorem 7.4 *$Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon)$ is holomorphic on the domain $\mathcal{D}^\epsilon$.* $\square$

We can obtain the automorphic properties of $Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon)$ in the same way using that for $\theta^{(2)}_L(\Omega)$ together with Theorem 6.8. Rather than do this explicitly, let us introduce a third variation of the Z-function, namely the *normalized partition function*

$$\hat{Z}^{(2)}_{V_L,\epsilon}(\tau_1,\tau_2,\epsilon)=\frac{Z^{(2)}_{V_L}(\tau_1,\tau_2,\epsilon)}{Z^{(2)}_{M^l}(\tau_1,\tau_2,\epsilon)}. \tag{147}$$

Bearing in mind the convention (124), what (130) says is that there is a commuting diagram of holomorphic maps

$$\begin{array}{ccc} \mathcal{D}^\epsilon & \xrightarrow{F^\epsilon} & \mathbb{H}_2 \\ \hat{Z}^{(2)}_{V_L,\epsilon} \searrow & & \swarrow \theta^{(2)}_L \\ & \mathbb{C} & \end{array} \tag{148}$$

Furthermore, the G-actions on the two functions in question are compatible. More precisely, if $\gamma\in G$ then we have

$$\begin{aligned}
\hat{Z}^{(2)}_{V_L,\epsilon}(\tau_1,\tau_2,\epsilon)|_{l/2}\,\gamma &= \hat{Z}^{(2)}_{V_L,\epsilon}(\gamma(\tau_1,\tau_2,\epsilon))\det(C\Omega+D)^{-l/2} \\
&= \theta^{(2)}_L(F^\epsilon(\gamma(\tau_1,\tau_2,\epsilon)))\det(C\Omega+D)^{-l/2} && ((148)) \\
&= \theta^{(2)}_L(\gamma(F^\epsilon(\tau_1,\tau_2,\epsilon)))\det(C\Omega+D)^{-l/2} && (\text{Theorem 2.3}) \\
&= \theta^{(2)}_L(\gamma\Omega)\det(C\Omega+D)^{-l/2}\ ] && ((124)) \\
&= \theta^{(2)}_L(\Omega)|_{l/2}\,\gamma. && (149)
\end{aligned}$$

For example, if the lattice L is *unimodular* as well as even then $\theta^{(2)}_L$ is a Siegel modular form of weight $l/2$ on the full group $Sp(4,\mathbb{Z})$. Then (149) informs us that

$$\hat{Z}^{(2)}_{V_L,\epsilon}(\tau_1,\tau_2,\epsilon)|_{l/2}\gamma=\hat{Z}^{(2)}_{V_L,\epsilon}(\tau_1,\tau_2,\epsilon),\ \gamma\in G,$$

i.e. $\hat{Z}^{(2)}_{V_L,\epsilon}(\tau_1,\tau_2,\epsilon)$ is automorphic of weight $l/2$ with respect to the group G.

8 Genus two partition function for free bosons in the ρ-formalism

8.1 The genus two partition function $Z_V^{(2)}(\tau, w, \rho)$

In this and the next Section we carry out an analysis of the partition function $Z_V^{(2)}(\tau, w, \rho)$ for Heisenberg free bosonic and lattice VOAs V in the ρ-formalism, which obtains when we consider sewing a twice-punctured torus to itself as reviewed in Section 2.3. The main results mirror those obtained in the ϵ-formalism. Recall that our definition (100) of the partition function in this case is

$$Z_{V,\rho}^{(2)}(\tau, w, \rho) = Z_V^{(2)}(\tau, w, \rho) = \sum_{n\geq 0} \sum_{u\in V_{[n]}} Z_V^{(1)}(u, \bar{u}, w, \tau)\rho^n. \tag{150}$$

Here, as before, u ranges over a basis of $V_{[n]}$ and $\bar{u}$ is the dual state with respect to the square-bracket LiZ metric. $Z^{(1)}(u, v, w, \tau)$ is the genus one 2-point function (90). Again (150) is multiplicative over V and independent of the choice of basis, and as long as we can choose a diagonal basis $\{u\}$ (e.g. the theory M for a free boson) then

$$Z_V^{(2)}(\tau, w, \rho) = \sum_{n\geq 0} \sum_{u\in V_{[n]}} \frac{Z_V^{(1)}(u, u, w, \tau)}{\langle u, u\rangle_{\mathrm{sq}}}\rho^n. \tag{151}$$

For one free boson, we choose the same diagonal basis (104) as before, so that these states $v = v(\lambda)$ may be indexed by unrestricted partitions λ. We have seen that λ determines a labelled set Φ_λ; what is also important in the present context is another labelled set $\Phi_{\lambda,2}$ defined to be the disjoint union of two copies of Φ_λ, call them $\Phi_\lambda^{(1)}$, $\Phi_\lambda^{(2)}$. Fix an identification $\iota : \Phi_\lambda^{(1)} \leftrightarrow \Phi_\lambda^{(2)}$.

The 2-point function $Z_M^{(1)}(v(\lambda), v(\lambda), w, \tau)$ was explicitly described in ([MT1], Corollary 1), where it was denoted $F_M(v, w_1, v, w_2; \tau)$. In what follows we continue to use the notation $w_{12} = w_1 - w_2 = w$, $w_{21} = w_2 - w_1 = -w$. Then we have (loc. cit.)

$$Z_M^{(1)}(v(\lambda), v(\lambda), w, \tau) = Z_M^{(1)}(\tau) \sum_{\phi\in F(\Phi_{\lambda,2})} \Gamma(\phi), \tag{152}$$

with

$$\Gamma(\phi, w, \tau) = \Gamma(\phi) = \prod_{\{r,s\}} \xi(r, s, w, \tau). \tag{153}$$

Furthermore, ϕ ranges over the elements of $F(\Phi_{\lambda,2})$ (fixed-point-free involutions in $\Sigma(\Phi_{\lambda,2})$), $\{r, s\}$ ranges over the orbits of ϕ on $\Phi_{\lambda,2}$, and

$$\xi(r, s, w, \tau) = \begin{cases} C(r, s, \tau), & \text{if } \{r, s\} \subseteq \Phi_\lambda^{(i)}, i = 1 \text{ or } 2, \\ D(r, s, w_{ij}, \tau) & \text{if } r \in \Phi_\lambda^{(i)}, s \in \Phi_\lambda^{(j)}, i \neq j. \end{cases}$$

Remark 8.1 *Note that ξ is well-defined since $D(r, s, w_{ij}, \tau) = D(s, r, w_{ji}, \tau)$.*

As usual, the fixed-point-free involution ϕ defines a complete matching μ_ϕ on the underlying labelled set $\Phi_{\lambda,2}$. In the ϵ-formalism we dealt with a pair of complete matchings based on Φ_λ; now we have just a single fixed-point-free involution acting on two copies of Φ_λ. However, we can supplement μ_ϕ with the 'canonical' matching defined by ι, which can also be considered as a fixed-point-free involution. Graphically we may represent μ_ι by a broken line, so that the analog of figure 5 is

$$\begin{array}{ccccc} \overset{r_1}{\bullet} & \xrightarrow{\phi} & \overset{s_1}{\bullet} & \overset{\iota}{--\rightarrow} & \overset{s_1'}{\bullet} \\ \overset{r_2}{\bullet} & \xrightarrow{\phi} & \overset{s_2}{\bullet} & \overset{\iota}{--\rightarrow} & \overset{s_2'}{\bullet} \\ \vdots & & \vdots & \vdots & \vdots \\ \overset{r_b}{\bullet} & \xrightarrow{\phi} & \overset{s_b}{\bullet} & \overset{\iota}{--\rightarrow} & \overset{s_b'}{\bullet} \end{array}$$

Fig. 7

where $\Phi_{\lambda,2} = \{r_1, ..., r_b\} = \{s_1, ..., s_b\}$ and $\iota : s \mapsto s'$ is the canonical label-preserving identification of the two copies of Φ_λ.

We create analogs of the chequered diagrams of Section 6.1, where now the cycles correspond to orbits of the cyclic group $\langle \iota\phi \rangle$ and the edges are not alternately labelled by integers $1, 2$ but rather by solid and broken lines corresponding to the action of the involutions ϕ and ι respectively. In addition to a positive integer, each node carries a label $a \in \{1, 2\}$ to indicate that it belongs to $\Phi_\lambda^{(a)}$, where the canonical involution satisfies $\iota : a \mapsto \bar{a}$ with convention (20). We call such objects *doubly-indexed chequered cycles (diagrams)*. Thus a doubly-indexed chequered cycle with four nodes looks as follows:

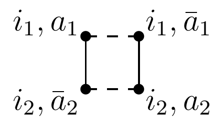

Fig. 8 A Doubly-Indexed Chequered Cycle

We have no need to consider all permutations of $\Phi_{\lambda,2}$; the only ones of relevance are those that commute with ι and preserve both $\Phi_\lambda^{(1)}$ and $\Phi_\lambda^{(2)}$. We denote this group, which is plainly isomorphic to $\Sigma(\Phi_\lambda)$, by Δ_λ. By definition, an automorphism of a chequered diagram D in the above sense is an element of Δ_λ which preserves edges and node labels.

For a chequered diagram D corresponding to the partition $\lambda = \{1^{e_1} ... p^{e_p}\}$ we set

$$\gamma(D) = \frac{\prod_{\{k,l\}} \xi(k,l,w,\tau)}{\prod(-i)^{e_i}} \rho^{\sum ie_i} \tag{154}$$

where $\{k,l\}$ ranges over the solid edges of D. We now have all the pieces assembled to copy the arguments of Section 6. First use the group Δ_λ in place of $\Sigma(\Phi_\lambda)$ to get the analog of Proposition 6.3, namely

$$Z_M^{(2)}(\tau,w,\rho) = Z_M^{(1)}(\tau) \sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|}, \tag{155}$$

the sum ranging over all chequered diagrams. The further analysis of this equality again proceeds as before, and we find that

$$Z_M^{(2)}(\tau,w,\rho) = Z_M^{(1)}(\tau) \prod_{\mathcal{R}} (1-\gamma(L))^{-1/2} \tag{156}$$

where L ranges over (oriented) rotationless chequered cycles $\mathcal{R}$.

Now introduce the infinite matrix $R^\vee = (R^\vee_{ab}(k,l))$, $k,l \geq 1, a,b \in \{1,2\}$, given by the block matrix

$$R^\vee(k,l) = -\frac{\rho^{(k+l)/2}}{\sqrt{kl}} \left[\begin{array}{cc} C(k,l,\tau) & D(k,l,\tau,w) \\ D(l,k,\tau,w) & C(k,l,\tau) \end{array} \right].$$

Compared to (37) then, we have $R^\vee_{ab}(k,l) = R_{a\bar{b}}(k,l)$.

The analog of the weight function ω (cf. (53)) is as follows: for a doubly-indexed chequered diagram D consisting of solid and broken edges as above

we set $\omega^\vee(D) = \prod \omega^\vee(E)$, the product running over all solid edges. Moreover for a solid edge E of type $\overset{k,a}{\bullet} \longrightarrow \overset{l,b}{\bullet}$ with nodes k, l lying in $\Phi_\lambda^{(a)}, \Phi_\lambda^{(b)}$ respectively, we set

$$\omega^\vee(E) = R^\vee_{a,b}(k,l) = (a,b)\text{-entry of } R^\vee(k,l).$$

The analog of Lemma 6.4 is

Lemma 8.2 $\omega^\vee(L) = \gamma(L)$.

Proof. From (154) it follows that for a chequered cycle L we have

$$\gamma(L) = \prod_{\{k,l\}} -\frac{\xi(k,l,w,\tau)\rho^{(k+l)/2}}{\sqrt{kl}}, \tag{157}$$

the product ranging over solid edges $\{k,l\}$ of L. This follows because in (154) each part of the relevant partition λ occurs twice. So to prove the Lemma it suffices to show that if the nodes k, l lie in $\Phi_\lambda^{(a)}, \Phi_\lambda^{(b)}$ respectively then the (a,b)-entry of $R^\vee(k,l)$ coincides with the corresponding factor of (157). This follows from our previous discussion together with Remark 8.1. $\square$

From Lemma 8.2 and (156) we obtain the analogs of Theorems 6.1 and 6.6, namely

Theorem 8.3 *Let M be the VOA of one free boson. Then*

$$Z_M^{(2)}(\tau,w,\rho) = \frac{Z_M^{(1)}(\tau)}{\det(1-R^\vee)^{1/2}}. \tag{158}$$

Theorem 8.4 *Let M be the VOA of one free boson. Then*

$$Z_M^{(2)}(\tau,w,\rho) = \frac{Z_M^{(1)}(\tau)}{\prod_{\mathcal{R}}(1-\omega^\vee(L))^{1/2}} \tag{159}$$

where L ranges over (oriented) rotationless chequered cycles $\mathcal{R}$.

We can re-formulate these results in a slightly different way using diagrams in which the canonical involution ι and broken edges are dispensed with, and utilizing a weight function ω associated to the matrix R rather

than $R^\vee$. First note that for a doubly-indexed chequered cycle L we have for suitable indices that

$$\omega^\vee(L) = R^\vee_{a_1a_2}(k_1,k_2)R^\vee_{\bar{a}_2a_3}(k_2,k_3)R^\vee_{\bar{a}_3a_4}(k_3,k_4)\ldots R^\vee_{\bar{a}_d\bar{a}_1}(k_d,k_1).$$

Since $R^\vee_{ab}(k,l) = R_{a\bar{b}}(k,l)$ then

$$\omega^\vee(L) = R_{a_1\bar{a}_2}(k_1,k_2)R_{\bar{a}_2\bar{a}_3}(k_2,k_3)R_{\bar{a}_3\bar{a}_4}(k_3,k_4)\ldots R_{\bar{a}_da_1}(k_d,k_1). \qquad (160)$$

It is apparent that we can interpret (160) as a weight function evaluated on the doubly-indexed cycles introduced in Section 3.3 with only regular (solid) edges and nodes indexed as before by a pair of positive integers k, a with $a \in \{1, 2\}$. Thus, a typical doubly-indexed cycle looks as follows:

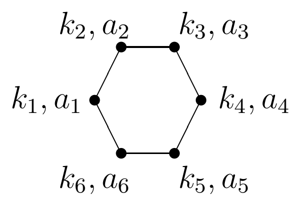

Fig. 9 Doubly-Indexed Cycle

If a doubly-indexed cycle has d nodes with consecutive edges labeled (k_1, k_2), (k_2, k_3), $\ldots(k_d, k_1)$ and corresponding second node indices $a_1, \ldots, a_d \in \{1, 2\}$, we set

$$\omega(L) = R_{a_1a_2}(k_1,k_2)R_{a_2a_3}(k_2,k_3)\ldots R_{a_da_1}(k_d,k_1). \qquad (161)$$

Our discussion shows that we may now reformulate Theorems 8.3 and 8.4 as follows:

Theorem 8.5 *Let M be the vertex operator algebra of one free boson. Then*

$$Z^{(2)}_M(\tau, w, \rho) = \frac{Z^{(1)}_M(\tau)}{\det(1-R)^{1/2}}. \qquad (162)$$

Theorem 8.6 *Let M be the vertex operator algebra of one free boson. Then*

$$Z^{(2)}_M(\tau, w, \rho) = \frac{Z^{(1)}_M(\tau)}{\prod_L (1-\omega(L))^{1/2}} \qquad (163)$$

where L ranges over (oriented) rotationless doubly-indexed cycles. $\square$

8.2 Holomorphic and modular-invariance properties

In Section 2.3 we reviewed the genus two ρ-sewing formalism and introduced the domain $\mathcal{D}^{\rho}$ which parameterizes the genus two surface. An immediate consequence of Theorem 2.4 is the following.

Theorem 8.7 *$Z_M^{(2)}(\tau, w, \rho)$ is holomorphic in $\mathcal{D}^{\rho}$.* $\square$

We next consider the invariance properties of the genus two partition function with respect to the action of the $\mathcal{D}^{\rho}$-preserving group Γ_1 reviewed in Section 2.3. As in Section 6.2, we again define a modular genus two partition function for two free bosons by

$$\begin{aligned} Z^{(2)}_{M^2,\mathrm{mod}}(\tau, w, \rho) &= Z^{(2)}_{M^2}(\tau, w, \rho) q^{-1/12} \\ &= \frac{1}{\eta(\tau)^2 \det(1-R)}. \end{aligned} \tag{164}$$

We then have a natural analog of Theorem 6.8:

Theorem 8.8 *If $\gamma \in \Gamma_1$ then*

$$Z^{(2)}_{M^2,\mathrm{mod}}(\tau, w, \rho)|_{-1}\gamma = \chi(\gamma) Z^{(2)}_{M^2,\mathrm{mod}}(\tau, w, \rho).$$

Corollary 8.9 *If $\gamma \in \Gamma_1$ with $Z^{(2)}_{M^{24},\mathrm{mod}} = (Z^{(2)}_{M^2,\mathrm{mod}})^{12}$ then*

$$Z^{(2)}_{M^{24},\mathrm{mod}}(\tau, w, \rho)|_{-12}\gamma = Z^{(2)}_{M^{24},\mathrm{mod}}(\tau, w, \rho).$$

Proof. We have to show that

$$Z^{(2)}_{M^2,\mathrm{mod}}(\gamma.(\tau, w, \rho)) \det(C\Omega + D) = \chi(\gamma) Z^{(2)}_{M^2,\mathrm{mod}}(\tau, w, \rho) \tag{165}$$

for $\gamma \in \Gamma_1$ where $\det(C\Omega_{11} + D) = c_1\Omega_{11} + d_1$. The proof is similar to that of Theorem 6.8. Consider the determinant formula (162). For $\gamma \in \Gamma_1$ define

$$R'_{ab}(k, l, \tau, w, \rho) = R_{ab}(k, l, \frac{a_1\tau + b_1}{c_1\tau + d_1}, \frac{w}{c_1\tau + d_1}, \frac{\rho}{(c_1\tau + d_1)^2})$$

following (45). We find from Section 6.3 of [MT2] that

$$\begin{aligned} 1 - R' &= 1 - R - \kappa\Delta \\ &= (1 - \kappa S).(1 - R), \end{aligned}$$

where

$$\begin{aligned}
\Delta_{ab}(k,l) &= \delta_{k1}\delta_{l1}, \\
\kappa &= \frac{\rho}{2\pi i}\frac{c_1}{c_1\tau+d_1}, \\
S_{ab}(k,l) &= \delta_{k1}\sum_{c\in\{1,2\}}((1-R)^{-1})_{cb}(1,l).
\end{aligned}$$

Since $\det(1-R)$ and $\det(1-R')$ are convergent on $\mathcal{D}^\rho$ we find

$$\det(1-R') = \det(1-\kappa S).\det(1-R).$$

Indexing the columns and rows by $(a,k) = (1,1),(2,1),\ldots(1,k),(2,k)\ldots$ and noting that $S_{1b}(k,l) = S_{2b}(k,l)$ we find that

$$\begin{aligned}
\det(1-\kappa S) &= \begin{vmatrix} 1-\kappa S_{11}(1,1) & -\kappa S_{12}(1,1) & -\kappa S_{11}(1,2) & \cdots \\ -\kappa S_{11}(1,1) & 1-\kappa S_{12}(1,1) & -\kappa S_{11}(1,2) & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \\
&= 1-\kappa S_{11}(1,1)-\kappa S_{12}(1,1), \\
&= 1-\kappa\sigma((1-R)^{-1})(1,1)).
\end{aligned}$$

Here and below, $\sigma(M)$ denotes the sum of the entries of a (finite) matrix M. Applying (40), it is clear that

$$\det(1-\kappa S) = \frac{c_1\Omega_{11}+d_1}{c_1\tau+d_1}.$$

The Theorem then follows from (121). $\square$

Remark 8.10 *$Z^{(2)}_{M^2,\mathrm{mod}}(\tau,w,\rho)$ can be trivially considered as function on the covering space $\hat{\mathcal{D}}^\rho$ discussed in Section 6.3 of ref. [MT2]. Then $Z^{(2)}_{M^2,\mathrm{mod}}(\tau,w,\rho)$ is modular with respect to $L = \hat{H}\Gamma_1$ with trivial invariance under the action of the Heisenberg group $\hat{H}$ (op. cite.).*

9 Genus two partition function for lattice theories in the ρ-formalism

Let L be an even lattice of rank l. The underlying Fock space for the corresponding lattice theory is given by (128), and the general shape of the

corresponding partition function is as in (150). We utilize the same basis for Fock space as before, namely $\{u \otimes e^{\alpha}\}$ where α ranges over L and u ranges over the usual orthogonal basis for M^l. From Lemma 4.3 and Corollary 4.4 we see that

$$Z_{V_L}^{(2)}(\tau, w, \rho) = \sum_{\alpha,\beta\in L}\sum_{n\geq 0}\sum_{u\in M^l_{[n]}} \frac{Z^{(1)}_{M^l\otimes e^{\beta}}(u\otimes e^{\alpha}, u\otimes e^{-\alpha}, w, \tau)}{\langle u, u\rangle_{\mathrm{sq}}}\rho^{n+(\alpha,\alpha)/2}. \tag{166}$$

The general shape of the 2-point function occurring in (166) is discussed extensively in [MT1]. By Proposition 1 (loc. cit.) it splits as a product

$$Z^{(1)}_{M^l\otimes e^{\beta}}(u\otimes e^{\alpha}, u\otimes e^{-\alpha}, w, \tau) = Q^{\alpha}_{M^l\otimes e^{\beta}}(u, u, w, \tau) Z^{(1)}_{M^l\otimes e^{\beta}}(e^{\alpha}, e^{-\alpha}, w, \tau), \tag{167}$$

where we have identified e^{α} with $\mathbf{1}\otimes e^{\alpha}$, and where $Q^{\alpha}_{M^l\otimes e^{\beta}}$ is a function[4] that we will shortly discuss in greater detail. In [MT1], Corollary 5 (cf. the Appendix to the present paper) we established also that

$$Z^{(1)}_{M^l\otimes e^{\beta}}(e^{\alpha}, e^{-\alpha}, w, \tau) = \epsilon(\alpha, -\alpha) q^{(\beta,\beta)/2}\frac{\exp((\beta,\alpha)w)}{K(w,\tau)^{(\alpha,\alpha)}} Z^{(1)}_{M^l}(\tau), \tag{168}$$

where as usual we are taking w in place of $z_{12} = z_1 - z_2$. With cocycle choice $\epsilon(\alpha, -\alpha) = (-1)^{(\alpha,\alpha)/2}$ (cf. Appendix) we may then rewrite (166) as

$$Z_{V_L}^{(2)}(\tau, w, \rho) = Z^{(1)}_{M^l}(\tau)\sum_{\alpha,\beta\in L}\exp\{\pi i[(\beta,\beta)\tau + 2(\alpha,\beta)\frac{w}{2\pi i} + \frac{(\alpha,\alpha)}{2\pi i}\log(\frac{-\rho}{K(w,\tau)^2})]\} \sum_{n\geq 0}\sum_{u\in M^l_{[n]}}\frac{Q^{\alpha}_{M^l\otimes e^{\beta}}(u, u, w, \tau)}{\langle u, u\rangle_{\mathrm{sq}}}\rho^{n}. \tag{169}$$

We note that this expression is, as it should be, independent of the choice of branch for the logarithm function. We are going to establish the *precise* analog of Theorem 7.1, to wit:

[4]Note: in [MT1] the functional dependence on α, here denoted by a superscript, was omitted.

Theorem 9.1 *We have*

$$Z^{(2)}_{V_L}(\tau, w, \rho) = Z^{(2)}_{M^l}(\tau, w, \rho)\theta^{(2)}_L(\Omega).$$

As in the case of the ϵ-formalism, we first handle the case of rank 1 lattices, then consider the general case. The inner double sum in (169) is the object which requires attention, and we can begin to deal with it along the lines of previous Sections. Namely, arguments that we have already used several times show that the double sum may be written in the form

$$\sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|} = \exp\left(\frac{1}{2}\sum_N \gamma(N)\right).$$

Here, D ranges over the doubly indexed chequered diagrams of Section 8, while N ranges over oriented, doubly indexed chequered diagrams which are connected. Leaving aside the definition of $\gamma(N)$ for now, we recognize as before that the piece involving only connected diagrams with no end nodes (aka doubly indexed chequered cycles) splits off as a factor. Apart from a $Z^{(1)}_M(\tau)$ term this factor is, of course, precisely the expression (155) for the free boson. With these observations, we see from (169) that the following holds:

$$\begin{aligned} \frac{Z^{(2)}_{V_L}(\tau, w, \rho)}{Z^{(2)}_M(\tau, w, \rho)} &= \\ \sum_{\alpha,\beta\in L} \exp\{\pi i[(\beta,\beta)\tau + 2(\alpha,\beta)\frac{w}{2\pi i} &+ \frac{(\alpha,\alpha)}{2\pi i}\log(\frac{-\rho}{K(w,\tau)^2}) \\ &+ \frac{1}{2\pi i}\sum_{N_{\alpha,\beta}}\gamma(N_{\alpha,\beta})]\}. \end{aligned} \tag{170}$$

The expression (131) for $\theta^{(2)}_L(\Omega)$ is invariant under interchange of α and β as well as replacing (α, β) by $-(\alpha, \beta)$, as noted in Remark 7.2. So to prove Theorem 9.1, we see from (170) that it is sufficient to establish that for each pair of lattice elements $\alpha, \beta \in L$, we have

$$\begin{aligned} (\beta,\beta)\Omega_{11} + 2(\alpha,\beta)\Omega_{12} + (\alpha,\alpha)\Omega_{22} &= \\ (\beta,\beta)\tau + 2(\alpha,\beta)\frac{w}{2\pi i} + \frac{(\alpha,\alpha)}{2\pi i}\log(\frac{-\rho}{K(w,\tau)^2}) \\ + \frac{1}{2\pi i}\sum_{N_{\alpha,\beta}}\gamma(N_{\alpha,\beta}). \end{aligned} \tag{171}$$

Recall the formulas for the Ω_{ij} (40-42). From now on we fix a pair of lattice elements α, β. In order to reconcile (171) with the formulas for the Ω_{ij}, we must carefully consider the expression $\sum_{N_{\alpha,\beta}} \gamma(N_{\alpha,\beta})$. To begin with, the $N_{\alpha,\beta}$ here essentially range over oriented chequered necklaces as used in Sections 3 and 8, except that in the present case the integer labelling of the end nodes is unrestricted (heretofore, it was required to be 1). The function γ is essentially (154), except that we also get contributions from the end nodes which are now present. Suppose that an end node has label $k \in \Phi^{(a)}, a \in \{1, 2\}$. Then according to Proposition 1 and display (45) of [MT1] (cf. the Appendix to the present paper), the contribution of the end node is equal to

$$\xi_{\alpha,\beta}(k, a, \tau, w, \rho) =$$
$$\begin{cases} \frac{\rho^{k/2}}{\sqrt{k}}(a, \delta_{k,1}\beta + C(k, 0, \tau)\alpha - D(k, 0, w, \tau)\alpha), & a = 1 \\ \frac{\rho^{k/2}}{\sqrt{k}}(a, \delta_{k,1}\beta - C(k, 0, \tau)\alpha + D(k, 0, -w, \tau)\alpha), & a = 2 \end{cases} \tag{172}$$

together with a contribution arising from the -1 in the denominator of (154) (we will come back to this point later). Using (cf. [MT1], displays (6), (11) and (12))

$$\begin{aligned} D(k, 0, -w, \tau) &= (-1)^{k+1} P_k(-w, \tau) = -P_k(w, \tau), \\ C(k, 0, \tau) &= (-1)^{k+1} E_k(\tau), \end{aligned}$$

we can combine the two possibilities in (172) thus (recalling that $E_k = 0$ for odd k):

$$\xi_{\alpha,\beta}(k, a, \tau, w, \rho) =$$
$$-\frac{\rho^{k/2}}{\sqrt{k}}(a, \delta_{k,1}\beta + (-1)^{(k+1)a}[P_k(w, \tau) - E_k(\tau)]\alpha). \tag{173}$$

Now consider an oriented chequered necklace $N = N_{\alpha,\beta}$ with a pair of end nodes labelled by $k \in \Phi^{(a)}$ and $l \in \Phi^{(b)}$ respectively. It looks like

$$\overset{k,a}{\bullet} \longrightarrow \bullet - \rightarrow - \ldots\ldots - \rightarrow - \bullet \longrightarrow \overset{l,b}{\bullet}$$

Fig. 10

From (173) we see that the total contribution of the end nodes to $\gamma(N)$ is

$$
\begin{aligned}
&-\xi_{\alpha,\beta}(k,a,\tau,w,\rho)\xi_{\alpha,\beta}(l,b,\tau,w,\rho) = \\
&-\frac{\rho^{(k+l)/2}}{\sqrt{kl}}(a,\delta_{k,1}\beta+(-1)^{(k+1)a}[P_k(w,\tau)-E_k(\tau)]\alpha) \\
&(a,\delta_{l,1}\beta+(-1)^{(l+1)b}[P_l(w,\tau)-E_l(\tau)]\alpha),
\end{aligned}
$$

where we note that a sign -1 arises from each *pair* of nodes, as follows from (154). Furthermore, if $N'_{\alpha,\beta}$ denotes the oriented chequered necklace from which the two end nodes and edges have been *removed* (we refer to these as *shortened* necklaces), then we have

$$
\begin{aligned}
\gamma(N_{\alpha,\beta}) &= -\xi_{\alpha,\beta}(k,a,\tau,w,\rho)\xi_{\alpha,\beta}(l,b,\tau,w,\rho)\gamma(N'_{\alpha,\beta}) \\
&= -\frac{\rho^{(k+l)/2}}{\sqrt{kl}}\gamma(N'_{\alpha,\beta})\{\delta_{k,1}\delta_{l,1}(\beta,\beta)+ \\
&\quad (-1)^{(k+1)a+(l+1)b}(\alpha,\alpha)[P_k(w,\tau)-E_k(\tau)][P_l(w,\tau)-E_l(\tau)]+ \\
&\quad (\alpha,\beta)[\delta_{k,1}(-1)^{(l+1)b}[P_l(w,\tau)-E_l(\tau)] \\
&\quad +\delta_{l,1}(-1)^{(k+1)a}[P_k(w,\tau)-E_k(\tau)]\}.
\end{aligned} \tag{174}
$$

We will now consider the terms corresponding to $(\beta,\beta),(\alpha,\beta)$ and (α,α) separately, and show that they are precisely the corresponding terms on each side of (171). This will complete the proof of Theorem 9.1 in the case of rank 1 lattices. From (174), a (β,β) term arises only if the end node weights k,l are both equal to 1, in which case we get

$$-\rho\gamma(N'_{\alpha,\beta}).$$

Then the total (β,β)-contribution to the right-hand-side of (171) is equal to

$$\tau-\frac{\rho}{2\pi i}\sum\gamma(N'_{\alpha,\beta}), \tag{175}$$

where the sum ranges over shortened necklaces with end nodes having weight 1. However, from [MT2] and also from Section 8 (esp. following (156)) it follows that this sum is nothing else than $\sigma((I-R)^{-1}(1,1))$. So we see from (171) that (175) coincides with Ω_{11} (cf. (40)) as required.

Next, from (174) we see that an (α,β)-contribution arises whenever at least one of the end nodes has label 1. If the labels of the end nodes are

unequal then the shortened necklace with the *opposite* orientation makes an equal contribution. The upshot is that we may assume that the end node to the right of the shortened necklace has label $l = 1 \in \Phi^{(\bar{b})}$, as long as we count accordingly. Then the contribution to the (α, β)-term from (174) is equal to

$$-2\rho^{1/2} \sum_{k \geq 1} \frac{\rho^{k/2}}{\sqrt{k}} (-1)^{(k+1)a+1} [P_k(w, \tau) - E_k(\tau)] \sum \gamma(N'_{\alpha,\beta}),$$

where the inner sum ranges over shortened necklaces with end nodes of weight $1 \in \Phi^{(\bar{b})}$ and $k \in \Phi^{(\bar{a})}$. In this case the discussion in Section 8 shows that

$$\begin{aligned} & \sum (-1)^{(k+1)a+1} \gamma(N'_{\alpha,\beta}) = \sum (-1)^{(k+1)a} R^{\vee}_{\bar{a},a_1}(k, k_1) \ldots R^{\vee}_{a_{d-1},\bar{b}}(k_{d-1}, 1) \\ = \; & \sum (-1)^{(k+1)(\bar{a}+1)+1} R_{\bar{a},\bar{a}_1}(k, k_1) \ldots R_{\bar{a}_{d-1},b}(k_{d-1}, 1) \end{aligned}$$

(summed over all indices $k_1, .., k_{d-1} \geq 1$ and $a, a_1, ..., a_{d-1}, b \in \{1, 2\}$)

$$\begin{aligned} = \; & \sum_{a,b=1}^{2} (-1)^{(k+1)(a+1)+1} (I - R)^{-1}_{ab}(k, 1) \\ = \; & \sum_{b=1}^{2} \{-(I - R)^{-1}_{1b}(k, 1) + (-1)^k (I - R)^{-1}_{2b}(k, 1)\} \\ = \; & \sigma\{(-1, (-1)^k)(I - R)^{-1}(k, 1)\}. \end{aligned}$$

So the contribution to the (α, β)-term from (174) is now seen to be equal to

$$\begin{aligned} & -2\rho^{1/2} \sum_{k \geq 1} \sigma\{\frac{\rho^{k/2}}{\sqrt{k}} [P_k(w, \tau) - E_k(\tau)](-1, (-1)^k)(I - R)^{-1}(k, 1)\} \\ = \; & -2\rho^{1/2} \sigma((b(I - R)^{-1})(1)), \end{aligned}$$

where b is defined by (43). Finally then, the total contribution to the (α, β) term on the right-hand-side of (171) is found, using (41), to be

$$2\frac{w}{2\pi i} - 2\frac{\rho^{1/2}}{2\pi i} \sigma(b(I - R)^{-1}(1)) = 2\Omega_{12},$$

as required.

It remains to deal with the (α,α) term, the details of which are very much along the lines as the case (α,β) just handled. A similar argument shows that the contribution to the (α,α)-term from (174) is equal to

$$-b(I-R)^{-1}\bar{b}^T,$$

so that the total contribution to the (α,α) term on the right-hand-side of (171) is

$$\frac{1}{2\pi i}\log\left(\frac{-\rho}{K(w,\tau)^2}\right)-\frac{1}{2\pi i}b(I-R)^{-1}\bar{b}^T=\Omega_{22},$$

as in (42). This finally completes the proof of Theorem 9.1 in the case of rank 1 lattices. As for the general case - we adopt the mercy rule by omitting details! The reader who has progressed this far will have no difficulty in dealing with the general case, which follows by combining the calculations in the rank 1 case just completed together with the method used in Section 7 to deal with a general lattice in the ϵ-formalism. □

Finally, the automorphic properties of $Z^{(2)}_{V_L}(\tau,w,\rho)$ can be analyzed much as in previous sections to find

Theorem 9.2 $Z^{(2)}_{V_L}(\tau,w,\rho)$ *is holomorphic on the domain* $\mathcal{D}^\rho$. □

We again introduce a normalized partition function

$$\hat{Z}^{(2)}_{V_L,\rho}(\tau,w,\rho)=\frac{Z^{(2)}_{V_L}(\tau,w,\rho)}{Z^{(2)}_{M^l}(\tau,w,\rho)},$$

so that by Theorem 9.1 there is a commuting diagram of holomorphic maps

$$\begin{array}{ccc}\mathcal{D}^\rho & \xrightarrow{F^\rho} & \mathbb{H}_2\\ \hat{Z}^{(2)}_{V_L,\rho}\searrow & & \swarrow\theta^{(2)}_L\\ & \mathbb{C} & \end{array}$$

Furthermore, the Γ_1-actions on the two functions are compatible with

$$\hat{Z}^{(2)}_{V_L,\rho}(\tau,w,\rho)|_{l/2}\,\gamma=\theta^{(2)}_L(\Omega)|_{l/2}\,\gamma, \tag{176}$$

for all $\gamma\in\Gamma_1$.

10 Comparison of partition functions in the two formalisms

In this section we consider the relationship between the genus two boson and lattice partition functions computed in the ϵ- and ρ- formalisms of the previous Sections. Although, for a given VOA, the partition functions enjoy many similar properties, we find that neither the original nor the modular partition functions (cf. Sections 6.2 and 8.2) are equal in the two formalisms. This result follows from an explicit computation of the partition functions in the neighborhood of a two tori degeneration point for two free bosons. It therefore follows that there is likewise no equality between the partition functions in the ϵ- and ρ- formalisms for a lattice VOA.

As reviewed in Section 2, there exists a 1-1 map between appropriate Γ_1-invariant neighborhoods of $\mathcal{D}^{\epsilon}$ and $\mathcal{D}^{\chi} \cup \mathcal{D}^{\chi}_0$ about any two tori degeneration point (cf. Theorem 2.7). We will make particular use of the explicit relationship between $(\tau_1, \tau_2, \epsilon)$ and (τ, w, χ) to $O(w^4)$ as follows:

Proposition 10.1 *For* $(\tau, w, \chi) \in \mathcal{D}^{\chi} \cup \mathcal{D}^{\chi}_0$,

$$\begin{aligned}
2\pi i\tau_1 &= 2\pi i\tau + \frac{1}{12}(1-4\chi)w^2 + \frac{1}{144}\, E_2(\tau)\,(1-4\chi)^2\, w^4 + O(w^6),\\
2\pi i\tau_2 &= \log(f(\chi)) + \frac{1}{12}\,(1-4\,\chi)^2\, E_4(\tau)\, w^4 + O(w^6),\\
\epsilon &= -w\sqrt{1-4\chi}(1+(1-4\chi)E_2(\tau)w^2) + O(w^5).
\end{aligned}$$

A proof of Proposition 10.1 appears in the Appendix.

Theorem 5.6 tells us that $Z^{(2)}_{V,\epsilon}(\tau_1,\tau_2,\epsilon)$ and $Z^{(2)}_{V,\rho}(\tau,w,\rho)$ agree at any two-tori degeneration point. In particular, for the VOA M^2 of two free bosons, we have

$$\begin{aligned}
&\lim_{\rho,w\to 0} Z^{(2)}_{M^2,\rho}(\tau,w,\rho) = Z^{(1)}_{M^2}(q)Z^{(1)}_{M^2}(f(\chi))\\
&= \lim_{\epsilon\to 0} Z^{(2)}_{M^2,\epsilon}(\tau_1,\tau_2,\epsilon) = Z^{(1)}_{M^2}(q_1)Z^{(1)}_{M^2}(q_2), \qquad (177)
\end{aligned}$$

with $Z^{(1)}_{M^2}(q) = q^{1/12}/\eta^2(q)$. However, away from a degeneration point, these partition functions are not equal:

Proposition 10.2 *For two free bosons in the neighborhood of a two-tori degeneration point we have*

$$\frac{Z^{(2)}_{M^2,\epsilon}(\tau_1,\tau_2,\epsilon)}{Z^{(2)}_{M^2,\rho}(\tau,w,\rho)} = 1+\frac{1}{144}(1-4\chi)w^2+O(w^4).$$

Proof. We consider $Z^{(2)}_{M^2,\epsilon}(\tau_1,\tau_2,\epsilon)$ to $O(\epsilon^2)$ and then apply Proposition 10.1 to $O(w^2)$ to find that

$$\begin{aligned}
Z^{(2)}_{M^2,\epsilon}(\tau_1,\tau_2,\epsilon) &= Z^{(1)}_{M^2}(q_1)Z^{(1)}_{M^2}(q_2)(1+E_2(\tau_1)E_2(\tau_2)\epsilon^2+O(\epsilon^4)) \\
&= Z^{(1)}_{M^2}\left(q(1+\frac{1}{12}(1-4\chi)w^2)\right)Z^{(1)}_{M^2}(f(\chi)) \\
&\quad \left(1+E_2(\tau)E_2(q=f(\chi))(1-4\chi)w^2+O(w^4)\right).
\end{aligned}$$

Recall that for modular form g_k of weight k on $SL(2,\mathbb{Z})$, the 'modular derivative'

$$\left(\frac{1}{2\pi i}\frac{d}{d\tau}+kE_2(\tau)\right)g_k(\tau) \tag{178}$$

is a modular form of weight $k+2$. Moreover the modular derivative of a cusp-form is again a cusp-form. In particular, the modular derivative of the cusp-form $\eta^{24}(\tau)$ of weight 12 vanishes since there are no nonzero cusp-forms of weight 14. Hence we find (as is well-known) that

$$\frac{1}{2\pi i}\frac{d}{d\tau}\eta^{-2}(\tau)=E_2(\tau)\eta^{-2}(\tau). \tag{179}$$

Therefore

$$Z^{(1)}_{M^2}(q(1+\frac{1}{12}(1-4\chi)w^2)) = Z^{(1)}_{M^2}(q)\left(1+\frac{1}{12}(E_2(\tau)+\frac{1}{12})(1-4\chi)w^2\right)+O(w^4).$$

Altogether, we therefore find

$$\frac{Z^{(2)}_{M^2,\epsilon}(\tau_1,\tau_2,\epsilon)}{Z^{(1)}_{M^2}(q)Z^{(1)}_{M^2}(f(\chi))} = 1+(E_2(\tau)G(\chi)+\frac{1}{144})(1-4\chi)w^2+O(w^4), \tag{180}$$

with $G(\chi)=E_2(q=f(\chi))-E_2(0)=E_2(f(\chi))+\frac{1}{12}$.

We next compute $Z^{(2)}_{M^2,\rho}(\tau,w,\rho)$ to $O(w^2)$ using (162). We expand R to $O(w^2)$ as described in Subsection 6.4 of [MT2] and in (197) of the Appendix. In particular, (177) implies

$$\det(I-R^{(0)})^{-1} = Z^{(1)}_{M^2}(f(\chi)), \tag{181}$$

with $R^{(0)}$ of (198). Then we find

$$\begin{aligned}\det(I-R)^{-1} &= Z^{(1)}_{M^2}(f(\chi))\det(I+w^2\chi E_2(\tau)(I-R^{(0)})^{-1}\Delta)+O(w^4)\\ &= Z^{(1)}_{M^2}(f(\chi))\left(1+w^2\chi E_2(\tau)\sigma((I-R^{(0)})^{-1}(1,1))\right)+O(w^4),\end{aligned}$$

with $\Delta_{ab}(k,l)=\delta_{k1}\delta_{l1}$. But applying (200) it follows that

$$\det(I-R)^{-1} = Z^{(1)}_{M^2}(f(\chi))\left(1+E_2(\tau)G(\chi)(1-4\chi)w^2\right)+O(w^4).$$

Hence we find

$$\frac{Z^{(2)}_{M^2,\rho}(\tau,w,\rho)}{Z^{(1)}_{M^2}(q)Z^{(1)}_{M^2}(f(\chi))} = 1+E_2(\tau)G(\chi)(1-4\chi)w^2+O(w^4). \tag{182}$$

Comparing to (180) we obtain the result. □

As described in Subsections 6.2 and 8.2, the bosonic genus two partition functions in both formalisms have enhanced modular properties on introducing a standard factor of $(q_1q_2)^{-c/24}$ in the ϵ-formalism and $q^{-c/24}$ in the ρ-formalism. It is therefore natural to also compare the modular partition functions $Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon)$ of (122) and $Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau,w,\rho)$ of (164). We firstly note that $Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon)q_2^{1/12}$ and $Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau,w,\rho)$ agree in the two tori degeneration limit. Furthermore, we find from above that

$$\frac{Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon)q_2^{1/12}}{Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau,w,\rho)} = 1+O(w^4).$$

(This result also follows from the fact that the ratio is invariant under the action of Γ_1 following Theorems 6.8 and 8.8). It follows that

$$\frac{Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon)}{Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau,w,\rho)} = f(\chi)^{-1/12}(1+O(w^4)). \tag{183}$$

Thus the modular partition functions do not agree in the two torus degeneration limit! Eqn. (183) suggests that a further modification of the modular partition functions be considered. We consider a possible further Γ_1-invariant factor of $f(\chi)^{-c/24}$ in the ρ-formalism and find

Proposition 10.3 *For two free bosons in the neighborhood of a two-tori degeneration point we have*

$$\frac{Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon)}{Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau,w,\rho)f(\chi)^{-1/12}} = 1+E_4(\tau)(\frac{73}{1440}+39H(\chi))\left(1-4\chi\right)^2 w^4+O(w^6),$$

where $H(\chi) = E_4(q=f(\chi)) - E_4(q=0) = E_4(f(\chi)) - \frac{1}{720}$.

Proof. We give a brief sketch of the proof. First consider $Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon)$ from (103) to $O(\epsilon^4)$ to find

$$\frac{1+E_2(\tau_1)E_2(\tau_2)\epsilon^2+\left(E_2(\tau_1)^2E_2(\tau_2)^2+54E_4(\tau_1)E_4(\tau_2)\right)\epsilon^4}{\eta(\tau_1)^2\eta(\tau_2)^2}+O(\epsilon^6). \quad (184)$$

We expand to $O(w^4)$ using Proposition 10.1 and use (179) and (192) to eventually find that

$$Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1,\tau_2,\epsilon) = \frac{1}{\eta(q)^2\eta(f(\chi))^2}[1+E_2\left(\tau\right)G(\chi)\left(1-4\chi\right)w^2$$
$$+\left(E_4\left(\tau\right)\left(\frac{73}{1440}+54H(\chi)+\frac{1}{2}\,G(\chi)\right)+E_2\left(\tau\right)^2G(\chi)^2\right)\left(1-4\,\chi\right)^2w^4]+O(w^6).$$

On the other hand, using the expansion of R in (197) of the Appendix and the methods of Subsection 5.2.2 of ref. [MT2], one eventually finds that

$$Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau,w,\rho)f(\chi)^{-1/12} = \frac{1}{\eta(q)^2\eta(f(\chi))^2}[1+E_2\left(\tau\right)G(\chi)\left(1-4\chi\right)w^2$$
$$+\left(E_4\left(\tau\right)\left(15H(\chi)+\frac{1}{2}\,G(\chi)\right)+E_2\left(\tau\right)^2G(\chi)^2\right)\left(1-4\,\chi\right)^2w^4]+O(w^6),$$

leading to the result. □

Remark 10.4 *Notice that $Z^{(2)}_{M^2,\mathrm{mod},\epsilon}$ and $Z^{(2)}_{M^2,\mathrm{mod},\rho}f(\chi)^{-1/12}$ differ to $O(w^4)$ in the leading χ^0 term. In particular, $R(k,l) = O(\chi)$ so that*

$$Z^{(2)}_{M^2,\mathrm{mod},\rho}(\tau, w, \rho)f(\chi)^{-1/12} = \frac{1}{\eta(q)^2\eta(f(\chi))^2} + O(\chi),$$

whereas (184) gives

$$Z^{(2)}_{M^2,\mathrm{mod},\epsilon}(\tau_1, \tau_2, \epsilon) = \frac{1 + \frac{73}{1440}E_4\left(\tau\right) w^4 + O(\chi)}{\eta(q)^2\eta(f(\chi))^2} + O(w^6).$$

11 Final Remarks

We have defined and explicitly calculated the genus two partition function for the Heisenberg free bosonic string and lattice VOAs in two separate sewing schemes, the ϵ- and ρ-formalisms. It is important to emphasize that our approach is constructively based only on the algebraic properties of a VOA. In particular, we have proved the holomorphy of the genus two partition functions on given domains $\mathcal{D}^\epsilon$ and $\mathcal{D}^\rho$ in the ϵ- and ρ-formalisms respectively, and have described modular properties for the modular partition functions. Lastly, we have shown that although the original (and modular) partition functions enjoy many properties in common between the two sewing schemes, for a given VOA they do not agree in the neighborhood of a two-tori degeneration point.

It is natural to try generalize these results to Riemann surfaces of higher genus g. We can inductively construct such surfaces from tori by sewing together lower genus surfaces and/or attaching handles [Y], [MT2]. In particular, we conjecture that for a general ϵ-sewing scheme joining together surfaces of genus g_1 and g_2, then Theorem 6.1 generalizes to

$$Z^{(g_1+g_2)}_{M,\epsilon} = Z^{(g_1)}_M Z^{(g_2)}_M (\det(1 - A_1A_2))^{-1/2},$$

where now A_1 and A_2 are the moment matrices of Section 4. in ref. [MT2] and $Z^{(g_a)}_M$ is the genus g_a partition function (in some chosen sewing scheme). Similarly, we conjecture that for a lattice VOA then Theorem 7.1 generalizes to

$$Z^{(g_1+g_2)}_{L,\epsilon} = Z^{(g_1+g_2)}_{M^l,\epsilon}\theta^{(g_1+g_2)}_L(\Omega),$$

where $\theta^{(g)}_L(\Omega)$ is the genus g Siegel theta function associated to L and Ω is the genus g period matrix. Similarly, we can consider attaching a handle to

a genus g surface in a general ρ-sewing scheme to conjecture the following generalizations of Theorems 8.5 and 9.1:

$$\begin{aligned} Z^{(g+1)}_{M,\rho} &= Z^{(g)}_{M}(\det(1-R))^{-1/2}, \\ Z^{(g+1)}_{L,\rho} &= Z^{(g+1)}_{M^l,\rho}\theta^{(g+1)}_{L}(\Omega), \end{aligned}$$

where R is the moment matrix of Section 5 in [MT2]. Indeed, these can be confirmed for $g = 0$ for the two sewing schemes described in Section 5.2. Then one obtains, respectively, $Z^{(1)}_{M}(q) = (\det(1-R))^{-1/2} = q^{1/24}/\eta(q)$ for $R = \mathrm{diag}(1-q, 1-q, 1-q^2, 1-q^2, \ldots)$ and $Z^{(1)}_{M,\rho} = (\det(1-R^{(0)}))^{-1/2} = Z^{(1)}_{M}(f(\chi))$ from (181).

In conclusion, let us briefly and heuristically sketch how these results compare to some related ideas in the physics and mathematics literature. There is a wealth of literature concerning the bosonic string e.g. [GSW], [P]. In particular, the conformal anomaly implies that the physically defined path integral partition function Z_{string} cannot be reduced to an integral over the moduli space $\mathcal{M}_g$ of a Riemann surface of genus g except for the 26 dimensional critical string where the anomaly vanishes. Furthermore, for the critical string, Belavin and Knizhnik argue that

$$Z_{\mathrm{string}} = \int\limits_{\mathcal{M}_g} |F|^2 \, d\mu,$$

where $d\mu$ denotes a natural volume form on $\mathcal{M}_g$ and F is holomorphic and non-vanishing on $\mathcal{M}_g$ [BK], [Kn]. They also claim that for $g \geq 2$, F is a global section for the line bundle $K \otimes \lambda^{-13}$ (where K is the canonical bundle and λ the Hodge bundle on $\mathcal{M}_g$) which is trivial by Mumford's theorem in algebraic geometry [Mu2]. In this identification, the λ^{-13} section is associated with 26 bosons, the K section with a $c = -26$ ghost system and the vanishing conformal anomaly to the vanishing first Chern class for $K \otimes \lambda^{-13}$ [N]. Recently, some of these ideas have also been rigorously proved for a zeta function regularized determinant of an appropriate Laplacian operator Δ_n [McT]. The genus two partition functions $Z^{(2)}_{M^2,\epsilon}(\tau_1, \tau_2, \epsilon)$ and $Z^{(2)}_{M^2,\rho}(\tau, w, \rho)$ constructed in this paper for a rank 2 Heisenberg VOA should correspond in these approaches to a local description of the holomorphic part of $\left(\frac{\det' \Delta_1}{\det N_1}\right)^{-1}$ of refs. [Kn], [McT], giving a local section of the line bundle λ^{-1}. As such, it

is therefore no surprise that $f_{\epsilon,\rho} = Z^{(2)}_{M^2,\epsilon}/Z^{(2)}_{M^2,\rho} \neq 1$ in the neighborhood of a two-tori degeneration point where the ratio of the two sections is a non-trivial transition function $f_{\epsilon,\rho}$.

In the case of a general rational conformal field theory, the conformal anomaly continues to obstruct the existence of a global partition function on moduli space for $g \geq 2$ [5]. However, *all* CFTs of a given central charge c are believed to share the same conformal anomaly e.g. [FS]. Thus, the identification of the normalized lattice partition functions of (8) presumably reflects the equality of the first Chern class of some bundle associated to a rank c lattice VOA to that for λ^{-c} with transition function $f_{\epsilon,\rho}^{c/2}$. It is interesting to note that even in the case of a unimodular lattice VOA with a unique conformal block ([MS], [TUY]) the genus two partition function can therefore only be described locally. It would obviously be extremely valuable to find a rigorous description of the relationship between the VOA approach described here and these related ideas in conformal field theory and algebraic geometry.

12 Appendix

12.1 A product formula

Here we continue the discussion initiated in Subsection 3.1, with a view to proving Proposition 3.1. Let $\mathcal{M}(I)$ be the (multiplicative semigroup generated by) the rotationless cycles in the symbols $x_i, i \in I$. There is an injection

$$\iota : \bigcup_{n\geq 0} C_n \longrightarrow \mathcal{M}(I) \tag{185}$$

defined as follows. If $(x) \in C_n$ has rotation group of order r then $r|n$ and there is a rotationless monomial y such that $x = y^r$. We then map $(x) \mapsto (y)^r$. It is readily verified that this is well-defined. In this way, each cycle is mapped to a power of a rotationless cycle in $\mathcal{M}(I)$. A typical element of $\mathcal{M}(I)$ is uniquely expressible in the form

$$p_1^{f_1} p_2^{f_2} \ldots p_k^{f_k} \tag{186}$$

where $p_1, \ldots, p_k$ are distinct rotationless cycles and $f_1, \ldots, f_k$ are non-negative integers. We call (186) the *reduced form* of an element in $\mathcal{M}(I)$. A general

[5] This observation appears to be contrary to the main result of [So2].

element of $\mathcal{M}(I)$ is then essentially a labelled graph, each of whose connected components are rotationless labelled polygons as discussed in Subsection 3.1.

Now consider a second finite set T together with a map

$$F : T \longrightarrow I. \tag{187}$$

Thus elements of I label elements of T via the map F. F induces a natural map

$$\overline{F} : \Sigma(T) \longrightarrow \mathcal{M}(I)$$

from the symmetric group $\Sigma(T)$ as follows. For an element $\tau \in \Sigma(T)$, write τ as a product of disjoint cycles $\tau = \sigma_1.\sigma_2 \ldots$. We set $\overline{F}(\tau) = \overline{F}(\sigma_1)\overline{F}(\sigma_2)...$, so it suffices to define $\overline{F}(\sigma)$ for a cycle $\sigma = (s_1 s_2 ...)$ with $s_1, s_2, ... \in T$. In this case we set

$$\overline{F}(\sigma) = \iota((x_{F(s_1)} x_{F(s_2)} ...))$$

where ι is as in (185). When written in the form (186), we call $\overline{F}(\tau)$ the *reduced F-form* of τ.

For $i \in I$, let $s_i = |F^{-1}(i)|$ be the number of elements in T with label i. So the number of elements in T is equal to $\sum_{i \in I} s_i$. We say that two elements $\tau_1, \tau_2 \in \Sigma(T)$ are F-equivalent if they have the same reduced F-form, i.e. $\overline{F}(\tau_1) = \overline{F}(\tau_2)$. We will show that each equivalence class contains the same number of elements. Precisely,

Lemma 12.1 *Each F-equivalence class contains precisely $\prod_{i \in I} s_i$ elements. In particular, the number of F-equivalence classes is $|T|!/\prod_{i \in I} s_i$.*

Proof. An element $\tau \in \Sigma(T)$ may be represented uniquely as

$$\begin{pmatrix} 0 & 1 & \cdots & M \\ \tau(0) & \tau(1) & \cdots & \tau(M) \end{pmatrix}$$

so that

$$\overline{F}(\tau) = \begin{pmatrix} F(0) & F(1) & \cdots & F(M) \\ F(\tau(0)) & F(\tau(1)) & \cdots & F(\tau(M)) \end{pmatrix}$$

with an obvious notation. Exactly s_i of the $\tau(j)$ satisfy

$$\overline{F}(\tau(j)) = x_i$$

so that there are $\prod_{i \in I} s_i$ choices of τ which have a given image under $\overline{F}$. The Lemma follows. $\square$

The next results employs notation introduced in Subsections 3.1 and 3.2.

Lemma 12.2 *We have*

$$(I - M_1M_2)^{-1}(1,1) = (1 - \sum_{L\in\mathcal{L}_{21}} \omega(L))^{-1}. \tag{188}$$

As before, the left-hand-side of (188) means $\sum_{n\geq 0}(M_1M_2)^n(1,1)$. It is a certain power series with entries being quasi-modular forms..

Proof of Lemma. We have

$$(M_1M_2)^n(1,1) = \sum M_1(1,k_1)M_2(k_1,k_2)\dots M_2(k_{2n-1},1) \tag{189}$$

where the sum ranges over all choices of positive integers $k_1, ..., k_{2n-1}$. Such a choice corresponds to a (isomorphism class of) chequered cycle L with $2n$ nodes and with at least one distinguished node, so that the left-hand-side of (188) is equal to

$$\sum_L \omega(L)$$

summed over all such L. We can formally write L as a product $L = L_1L_2...L_p$ where each $L_i \in \mathcal{L}_{21}$. This indicates that L has p distinguished nodes and that the L_i are the edges of L between consecutive distinguished nodes, which can be naturally thought of as chequered cycles in $\mathcal{L}_{21}$. Note that in the representation of L as such a product, the L_i do not commute unless they are equal, moreover ω is multiplicative. Then

$$(I - M_1M_2)^{-1}(1,1) = \sum_{L_i\in\mathcal{L}_{21}} \omega(L_1...L_p) = (1 - \sum_{L\in\mathcal{L}_{21}} \omega(L))^{-1}$$

as required. $\square$

Proposition 12.3 *We have*

$$(I - M_1M_2)^{-1}(1,1) = \prod_{L\in\mathcal{R}_{21}} (1 - \omega(L))^{-1} \tag{190}$$

Proof. By Lemma 12.2 we have

$$(I - M_1M_2)^{-1}(1,1) = \sum m(e_1, ..., e_k)\omega(L_1)^{e_1}...\omega(L_k)^{e_k} \tag{191}$$

where the sum ranges over distinct elements $L_1, ...L_k$ of $\mathcal{L}_{21}$ and all k-tuples of non-negative integers $e_1, ..., e_k$, and where the multiplicity is

$$m(e_1, ..., e_k) = \frac{(\sum_i e_i)!}{\prod_i(e_i!)}.$$

Let S be the set consisting of e_i copies of $L_i, 1 \leq i \leq k$, let I be the integers between 1 and k, and let $F : S \longrightarrow I$ be the obvious labelling map. A reduced F-form is then an element of $\mathcal{M}(I)$ where the variables x_i are now the L_i. The free generators of $\mathcal{M}(I)$, i.e. rotationless cycles in the x_i, are naturally identified *precisely* with the elements of $\mathcal{R}_{21}$, and Lemma 12.1 implies that each element of $\mathcal{M}(I)$ corresponds to just one term under the summation in (191). Eqn.(190) follows immediately from this and the multiplicativity of ω, and the Proposition is proved. $\square$

12.2 Invertibility about a two tori degeneration point

Here we prove Proposition 10.1 describing the relationship to $O(w^4)$ between $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^\epsilon$ and $(\tau, w, \chi) \in \mathcal{D}^\chi \cup \mathcal{D}^\chi_0$ in the neighborhood of a two tori degeneration point $\rho, w \to 0$ for fixed $\chi = -\rho/w^2$. We first expand Ω_{ij} to $O(\epsilon^4)$ using Theorem 2.2 to obtain

$$\begin{aligned}
2\,\pi i \Omega_{11} &= 2\,\pi i\,\tau_1 + E_2\left(\tau_2\right)\epsilon^2 + E_2\left(\tau_1\right)E_2\left(\tau_2\right)^2\epsilon^4 + O(\epsilon^6),\\
2\,\pi i \Omega_{12} &= -\epsilon - E_2\left(\tau_1\right)E_2\left(\tau_2\right)\epsilon^3 + O(\epsilon^5),\\
2\,\pi i \Omega_{22} &= 2\,\pi i\,\tau_2 + E_2\left(\tau_1\right)\epsilon^2 + E_2\left(\tau_1\right)^2E_2\left(\tau_2\right)\epsilon^4 + O(\epsilon^6).
\end{aligned}$$

Making use of the identity (cf. (178))

$$\frac{1}{2\,\pi i}\frac{d}{d\tau}E_2(\tau) = 5E_4(\tau) - E_2(\tau)^2, \tag{192}$$

it is straightforward to invert $\Omega_{ij}(\tau_1, \tau_2, \epsilon)$ to find

Lemma 12.4 *In the neighborhood of a two tori degeneration point $\Omega_{12} = 0$ of $\Omega \in \mathbb{H}_2$ we have*

$$\begin{aligned}
2\,\pi i\,\tau_1 &= 2\,\pi i \Omega_{11} - E_2\left(\Omega_{22}\right)r^2 + 5E_2\left(\Omega_{11}\right)E_4\left(\Omega_{22}\right)r^4 + O(r^6),\\
\epsilon &= -r + E_2\left(\Omega_{11}\right)E_2\left(\Omega_{22}\right)r^3 + O(r^5),\\
2\,\pi i\,\tau_2 &= 2\,\pi i \Omega_{22} - E_2\left(\Omega_{11}\right)r^2 + 5E_2\left(\Omega_{22}\right)E_4\left(\Omega_{11}\right)r^4 + O(r^6),
\end{aligned}$$

where $r = 2\,\pi i \Omega_{12}$. $\square$

We next obtain $\Omega_{ij}(\tau, w, \chi)$ to $O(w^4)$ in the neighborhood of a two tori degeneration point:

Proposition 12.5 *For* $(\tau, w, \chi) \in \mathcal{D}^\chi \cup \mathcal{D}_0^\chi$ *we have*

$$2\pi i\Omega_{11} = 2\pi i\tau + (1-4\chi)G(\chi)w^2 + (1-4\chi)^2 G(\chi)^2 E_2(\tau)\, w^4 + O(w^6), \tag{193}$$

$$2\pi i\Omega_{12} = w\sqrt{1-4\chi}(1+(1-4\chi)G(\chi)E_2(\tau)w^2) + O(w^5), \tag{194}$$

$$\begin{aligned} 2\pi i\Omega_{22} &= \log f(\chi) + (1-4\chi)E_2(\tau)w^2 \\ &\quad + (1-4\,\chi)^2 \left(G(\chi)E_2(\tau)^2 + \frac{1}{2}E_4(\tau) \right) w^4 + O(w^6), \end{aligned} \tag{195}$$

where

$$G(\chi) = \frac{1}{12} + E_2(q = f(\chi)), \tag{196}$$

and $f(\chi)$ *is the Catalan series (48).*

Proof. Ω_{12} is described in Proposition 13 of [MT2], whereas Ω_{11} and Ω_{22} are described there only to order $O(w^2)$. The $O(w^4)$ terms are calculated similarly, as follows. From (9) and (14) we have

$$\begin{aligned} P_1(\tau, w) &= \frac{1}{w}(1 - E_2(\tau)w^2 - E_4(\tau)w^4 + O(w^6)), \\ P_2(\tau, w) &= \frac{1}{w^2}(1 + E_2(\tau)w^2 + 3E_4(\tau)w^4 + O(w^6)), \\ P_3(\tau, w) &= \frac{1}{w^3}(1 - 3E_4(\tau)w^4 + O(w^6)), \\ P_4(\tau, w) &= \frac{1}{w^4}(1 + E_4(\tau)w^4 + O(w^6)), \end{aligned}$$

and $P_n(\tau, w) = w^{-n}(1 + O(w^6))$ for $n > 4$. Then $R(k, l)$ of (37) gives

$$R(k, l) = R^{(0)}(k, l) + R^{(2)}(k, l)w^2 + R^{(4)}(k, l)w^4 + O(w^6), \tag{197}$$

where

$$R^{(0)}(k, l) = \frac{(-\chi)^{(k+l)/2}}{\sqrt{kl}} \frac{(k+l-1)!}{(k-1)!(l-1)!} \left[\begin{array}{ll} (-1)^k & 0 \\ 0 & (-1)^l \end{array} \right]. \tag{198}$$

$R^{(0)}$ is associated with the self-sewing of a sphere to form a torus described in Subsections 5.2.2 and 6.4 of [MT2]. $R^{(2)}(k, l)$ is given by

$$R^{(2)}(k, l) = \chi E_2(\tau)\Delta(k, l), \quad \Delta(k, l) = \delta_{k1}\delta_{l1} \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right], \tag{199}$$

whereas $R^{(4)}(k,l)=0$ for $k+l>4$ with non-zero entries

$$\begin{aligned}
R^{(4)}(1,1) &= 3\chi E_4(\tau)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \\
R^{(4)}(1,2) &= -R^{(4)}(2,1) = 3\sqrt{2}(-\chi)^{3/2}E_4(\tau)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
R^{(4)}(1,3) &= R^{(4)}(3,1) = \sqrt{3}\chi^2 E_4(\tau)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \\
R^{(4)}(2,2) &= 3\chi^2 E_4(\tau)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\end{aligned}$$

We find for b of (43) that

$$b(k) = b^{(0)}(k) + b^{(2)}(k)w^2 + b^{(4)}(k)w^4 + O(w^6),$$

with

$$\begin{aligned}
b^{(0)}(k) &= \frac{(-\chi)^{k/2}}{\sqrt{k}}[-1,(-1)^k], \\
b^{(2)}(k) &= -\delta_{k1}E_2(q)b^{(0)}(1),
\end{aligned}$$

and $b^{(4)}(k)=0$ for $k>3$ with non-zero entries

$$\begin{aligned}
b^{(4)}(1) &= -E_4(\tau)b^{(0)}(1), \\
b^{(4)}(2) &= 3E_4(\tau)b^{(0)}(2), \\
b^{(4)}(3) &= -3E_4(\tau)b^{(0)}(3).
\end{aligned}$$

It is convenient to define

$$T = (I-R)^{-1} = \sum_{n\geq 0} R^n = T^{(0)} + T^{(2)}w^2 + T^{(4)}w^4 + O(w^6),$$

with

$$\begin{aligned}
T^{(0)} &= (I-R^{(0)})^{-1}, \\
T^{(2)} &= T^{(0)}R^{(2)}T^{(0)}, \\
T^{(4)} &= T^{(0)}R^{(2)}T^{(0)}R^{(2)}T^{(0)} + T^{(0)}R^{(4)}T^{(0)}.
\end{aligned}$$

We next recall from Proposition 10 and (170) of op.cite. that

$$\chi\sigma(T^{(0)}(1,1)) = \chi\sigma((I-R^{(0)})^{-1}(1,1)) = (1-4\chi)G(\chi). \tag{200}$$

We also find that

$$(b^{(0)}T^{(0)})_a(k) = (T^{(0)}\bar{b}^{(0)T})_a(k) = (-1)^k(-1)^{(k+1)a}.\frac{(-\chi)^{k/2}}{\sqrt{k}}\sum_{n\geq 1} S_{n,k}(\chi),$$

where $S_{1,k}(\chi) = 1$ and

$$\begin{aligned} S_{n,k}(\chi) &= \sum_{k_{n-1},\ldots k_1\geq 1} \chi^{k_{n-1}+\ldots+k_1}\binom{k+k_{n-1}-1}{k_{n-1}}\binom{k_{n-1}+k_{n-2}-1}{k_{n-2}} \\ &\quad \ldots\binom{k_2+k_1-1}{k_1}, \end{aligned} \tag{201}$$

for $n > 1$. But (129) of op.cite. states that

$$\sum_{n\geq 1} S_{n,k}(\chi) = (1+f(\chi))^k.$$

Hence we find

$$\frac{(-\chi)^{k/2}}{\sqrt{k}}(b^{(0)}T^{(0)})_a(k) = \frac{(-\chi)^{k/2}}{\sqrt{k}}(T^{(0)}\bar{b}^{(0)T})_a(k) = (-1)^{(k+1)a}.\frac{X(\chi)^k}{k}, \tag{202}$$

where for later convenience we have defined

$$X = X(\chi) = \chi(1+f(\chi)).$$

We also note that

$$(1-2X)^2 = 1-4\chi. \tag{203}$$

We may then compute Ω_{ij} to $O(w^4)$ employing the identities (200), (202) and (203) as follows. We begin with Ω_{11} of (40) which to $O(w^4)$ is given by

$$2\pi i\Omega_{11} = 2\pi i\tau + w^2\chi\sigma(T^{(0)}(1,1)) + w^4\chi\sigma(T^{(2)}(1,1)) + O(w^6).$$

Using (199) we find

$$\begin{aligned} \chi\sigma(T^{(2)}(1,1)) &= E_2(\tau)\chi^2\sigma(T^{(0)}\Delta T^{(0)})(1,1) \\ &= E_2(\tau)(\chi\sigma(T^{(0)}(1,1)))^2 \\ &= E_2(\tau)(1-4\chi)^2 G(\chi)^2, \end{aligned}$$

on applying (200). Hence (193) follows.

Expanding Ω_{22} of (42) to $O(w^4)$ we find

$$\begin{aligned}
2\pi i\Omega_{22} &= \log\chi - b^{(0)}T^{(0)}\bar{b}^{(0)T} + \left(E_2(\tau) - b^{(0)}T^{(2)}\bar{b}^{(0)T} - 2b^{(2)}T^{(0)}\bar{b}^{(0)T}\right)w^2 \\
&\quad +(\frac{1}{2}E_4(\tau) - b^{(0)}T^{(4)}\bar{b}^{(0)T} - b^{(2)}T^{(0)}\bar{b}^{(2)T} - 2b^{(2)}T^{(2)}\bar{b}^{(0)T} \\
&\quad -2b^{(4)}T^{(0)}\bar{b}^{(0)T})w^4 + O(w^6).
\end{aligned}$$

Proposition 9 of op.cite. states that

$$\log f(\chi) = \log\chi - b^{(0)}T^{(0)}\bar{b}^{(0)T}.$$

By applying (200) and (202) we obtain

$$\begin{aligned}
-b^{(0)}T^{(2)}\bar{b}^{(0)T} &= E_2(\tau)\left[(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1)\right]^2 = 4E_2(\tau)X^2, \\
-2b^{(2)}T^{(2)}\bar{b}^{(0)T} &= -2E_2(\tau)(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1) = -4E_2(\tau)X.
\end{aligned}$$

Thus the $O(w^2)$ term of $2\pi i\Omega_{22}$ is found using (203).

Similarly, one finds that

$$\begin{aligned}
-b^{(0)}T^{(4)}\bar{b}^{(0)T} &= E_2(\tau)^2\chi\sigma(T^{(0)})(1,1)\left[(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1)\right]^2 \\
&\quad +3E_4(\tau)\sum_{a=1,2}\left[(-\chi)^{1/2}(b^{(0)}T^{(0)})_a(1)\right]^2 \\
&\quad -6E_4(\tau)(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1).\frac{(-\chi)^1}{\sqrt{2}}\sigma(b^{(0)}T^{(0)})(2) \\
&\quad +6E_4(\tau)(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1)\frac{(-\chi)^{3/2}}{\sqrt{3}}\sigma(b^{(0)}T^{(0)})(3) \\
&= 4E_2(\tau)^2(1-4\chi)G(\chi)X^2 + E_4(\tau)(6X^2 - 12X^3 + 8X^4)
\end{aligned}$$

$$\begin{aligned}
-b^{(2)}T^{(0)}\bar{b}^{(2)T} &= E_2(\tau)^2\chi\sigma(T^{(0)})(1,1) = E_2(\tau)^2(1-4\chi)G(\chi), \\
-2b^{(2)}T^{(2)}\bar{b}^{(0)T} &= -2E_2(q)^2\chi\sigma(T^{(0)})(1,1)(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1) \\
&= -4E_2(q)^2(1-4\chi)G(\chi)X, \\
-2b^{(4)}T^{(0)}\bar{b}^{(0)T} &= -2E_4(\tau)[(-\chi)^{1/2}\sigma(b^{(0)}T^{(0)})(1) - 3\frac{(-\chi)^1}{\sqrt{2}}\sigma(b^{(0)}T^{(0)})(2) \\
&\quad +3\frac{(-\chi)^{3/2}}{\sqrt{3}}\sigma(b^{(0)}T^{(0)})(3)] \\
&= E_4(\tau)(-4X + 6X^2 - 4X^3).
\end{aligned}$$

Combining, we find the coefficient of $E_2(\tau)^2(1-4\chi)G(\chi)w^4$ in Ω_{22} is $(1-2X)^2 = 1-4\chi$ whereas the coefficient of $E_4(\tau)\,w^4$ is

$$\frac{1}{2} + 6X^2 - 12X^3 + 8X^4 - 4X + 6X^2 - 4X^3 = \frac{1}{2}(1-2X)^4 = \frac{1}{2}(1-4\chi)^2.$$

Hence (195) follows. $\square$

We next combine Lemma 12.4 and Proposition 12.5 to prove Proposition 10.1. Thus $2\,\pi i\,\tau_1$ is given by

$$\begin{aligned}
& 2\,\pi i\Omega_{11} - E_2(\Omega_{22})(2\pi i\Omega_{12})^2 + 5E_2(\Omega_{11})\,E_4(\Omega_{22})(2\pi i\Omega_{12})^4 + O(\Omega_{12}^6) \\
= \;& 2\pi i\tau + (1-4\chi)G(\chi)w^2 + (1-4\chi)^2\,G(\chi)^2E_2(\tau)\,w^4 \\
& - \left[E_2(f) + (5E_4(f) - E_2(f)^2)(1-4\chi)E_2(\tau)w^2\right] \\
& .w^2(1-4\chi)(1+2(1-4\chi)G(\chi)E_2(\tau)w^2) + \\
& +5E_2(\tau)\,E_4(f)\,(1-4\chi)^2w^4 + O(w^6) \\
= \;& 2\pi i\tau + [G(\chi) - E_2(f)]\,(1-4\chi)w^2,
\end{aligned}$$

where $E_2(f) = E_2(q = f(\chi))$ and $E_4(f) = E_4(q = f(\chi))$. Applying (196) we find

$$2\,\pi i\,\tau_1 = 2\pi i\tau + \frac{1}{12}(1-4\chi)w^2 + \frac{1}{144}E_2(\tau)(1-4\chi)^2w^4 + O(w^6).$$

Similarly

$$\begin{aligned}
\epsilon \;&=\; -2\pi i\Omega_{12} + E_2(\Omega_{11})\,E_2(\Omega_{22})(2\pi i\Omega_{12})^3 + O(\Omega_{12}^5) \\
&=\; -w\sqrt{1-4\chi}(1 + (1-4\chi)\,[G(\chi) - E_2(f)]\,E_2(\tau)w^2) + O(w^5).
\end{aligned}$$

Finally, $2\,\pi i\,\tau_2$ is given by

$$\begin{aligned}
& 2\,\pi i\Omega_{22} - E_2(\Omega_{11})(2\pi i\Omega_{12})^2 + 5E_2(\Omega_{22})\,E_4(\Omega_{11})(2\pi i\Omega_{12})^4 + O(\Omega_{12}^6) \\
= \;& \log f(\chi) + (1-4\chi)E_2(\tau)w^2 + (1-4\,\chi)^2\left(G(\chi)E_2(\tau)^2 + \frac{1}{2}E_4(\tau)\right)w^4 \\
& - \left[E_2(\tau) + (5E_4(\tau) - E_2(\tau)^2)(1-4\chi)G(\chi)w^2\right] \\
& .w^2(1-4\chi)(1+2(1-4\chi)G(\chi)E_2(\tau)w^2) \\
& +5E_2(f)\,E_4(\tau)\,(1-4\chi)^2w^4 + O(w^6) \\
= \;& \log f(\chi) + \left(\frac{1}{2} + 5[E_2(f) - G(\chi)]\right)E_4(\tau)\,(1-4\chi)^2w^4 + O(w^6).
\end{aligned}$$

Thus Proposition 10.1 is proved. $\square$

12.3 Corrections

We list here some corrections to [MT1] that we needed above. All references below are to [MT1].

(a) Display (27) should read

$$\epsilon(\alpha,-\alpha)=\epsilon(\alpha,\alpha)=(-1)^{(\alpha,\alpha)/2}. \tag{204}$$

(b) Display (45) should read

$$\gamma(\Xi)=(a,\delta_{r,1}\beta+C(r,0,\tau))\alpha_k+\sum_{l\neq k}D(r,0,z_{kl},\tau). \tag{205}$$

(c) As a result of (a), displays (79) and (80) are modified and now read

$$\begin{aligned} F_N(e^\alpha,z_1;e^{-\alpha},z_2;q) &= \epsilon(\alpha,-\alpha)\frac{q^{(\beta,\beta)/2}}{\eta^l(\tau)}\frac{\exp((\beta,\alpha)z_{12})}{K(z_{12},\tau)^{(\alpha,\alpha)}}, &(206)\\ F_{V_L}(e^\alpha,z_1;e^{-\alpha},z_2;q) &= \epsilon(\alpha,-\alpha)\frac{1}{\eta^l(\tau)}\frac{\Theta_{\alpha,L}(\tau,z_{12}/2\pi i)}{K(z_{12},\tau)^{(\alpha,\alpha)}}. &(207)\end{aligned}$$

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